ETF2700/ETF5970 Mathematics for Business

Lecture 9

Monash Business School, Monash University, Australia

Outline

Weeks 4-8:

- Calculus: differentiation and integration
- Optimization

This week (an introduction to investment):

- Basic theory of interest
- Cash flow stream: Sequences and series
- Present value techniques and applications

Questionnaire

Suppose that you are the winner of a lottery of 1 million dollars (after-tax), and you have two options:

1) Receive the money right now
2) Receive the same amount of money one year later

Which option would you choose?

A financial economist assumes you prefer the 1st option

If you receive the money right now, and deposit it into your bank account, one year later you will have more than $1m

1) You will receive **interest** for holding the deposit in your account. Your wealth will be more than $1m in one year
2) Present value of $1m in one year is less than $1m

Discount

Suppose an amount $A next year has a present value

$$A \cdot \frac{1}{1+r} < A$$

for some **discount factor** $\frac{1}{1+r} < 1$.

- $P = A/(1+r)$ is worth of $P(1+r)$ next year
- Often take $r > 0$ as the (annual) **interest rate**

Example

For $r = 2\%$, an amount of $1000 dollars in one year has a present value:

$$\frac{1000}{1+2\%} = \frac{1000}{1.02} \approx 980.39$$

If you receive $1m win right now, and deposit it into your bank account, one year later you will have more than $1m

Compound interest

Suppose the annual interest rate **from the bank** is $r > 0$, and interests are paid at m equally-spaced periods.

- the interest rate for each deposit period is $\frac{r}{m}$
- interest obtained from the 1st deposit period will earn interest in future:

$$P \to \underbrace{P\left(1+\frac{r}{m}\right)}_{1st \text{ period}} \to \underbrace{P\left(1+\frac{r}{m}\right)^2}_{2nd \text{ period}} \cdots \to \underbrace{P\left(1+\frac{r}{m}\right)^m}_{m\text{th period}}$$

- Shall we take an effective rate $r_{\text{eff}} > 0$ such that

$$P\left(1+\frac{r}{m}\right)^m \cdot \frac{1}{1+r_{\text{eff}}} = P?$$

Effective interest rate is the solution to

$$\left(1+\frac{r}{m}\right)^m \cdot \frac{1}{1+r_{\text{eff}}} = 1, \quad \text{for given values of } m \text{ and } r.$$

Effective rate

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

where r is therefore usually called the **nominal** rate.

Continuously compounded interest rate

As $m \to \infty$ (compound more and more frequently),

$$\left(1 + \frac{r}{m}\right)^m - 1 \to e^r - 1$$

Interest at different frequencies

Suppose that the annual interest rate is r

- If the interest is paid on m equally spaced periods, the annual **effective rate** is $r_{\text{eff}} = [1 + r/m]^m - 1$
- If interest is paid annually, the T-year effective rate (compounded for T years)

$$r_{\text{eff}} = (1 + r)^T - 1$$

Example

A deposit of \$5000 is put into an account earning interest at the annual rate of 9%, with interest paid quarterly. How much will there be in the account after 8 years.
With $r = 9\%$, $m = 4$, $k = 8m = 32$ and $P = 5000$, we will have

$$P \cdot \left(1 + \frac{r}{m}\right)^k = 5000 \cdot \left(1 + \frac{0.09}{4}\right)^{32} \approx 10190.52$$

Example

When investing in a savings account, which of the following offers is better?

- 5.9% with interest paid quarterly, or
- 6% with interest paid twice a year?

The corresponding annual effective rates are

$$r_{\text{eff},1} = (1 + 0.059/4)^4 - 1 \approx 0.0603 = 6.03\%$$
$$r_{\text{eff},2} = (1 + 0.06/2)^2 - 1 \approx 0.0609 = 6.09\%$$

The second offer is therefore better.

Investment project

We can use the discounting technique to

- compare financial values at different time periods;
- use effective rates; and
- help us evaluate investment project(s)

An investment project usually involves:

- an initial investment $x_0 < 0$: A negative cash flow
- cash flow x_t at (equally-spaced) periods $t = 1, 2, \ldots, T$
- the project may be operating forever: $T = \infty$

Mathematics in the evaluation of an investment project

From the investor's view, we have

- an initial investment x_0: out-flow cash
- in-flow cash amounts: x_t at time $t = 1, 2, \ldots, T$,

An investment project is often expressed by cash flow stream:

- Finite stream: $(x_0, x_1, x_2, \ldots, x_T)$
- Infinite stream: $(x_0, x_1, x_2, \ldots)$

Example: An investment project

Consider an investment project as follows:

- initial investment of 3 thousand dollars: $x_0 = -3$
- receive 2 thousand dollars in each of the next 2 months

$$x_1 = 2, \quad \text{and } x_2 = 2$$

The cash flow steam (in thousand \$) is: $(-3, 2, 2)$

Discounted cash flows

Suppose that the annual interest rate is $r = 6\%$.

- **monthly** interest rate is $r/12 = 0.5\%$
- discounted value of the cash flow from the 1st month:

$$\frac{x_1}{1 + r/12} = \frac{2}{1 + 0.5\%}$$

- discounted value of the cash flow from the 2nd month:

$$\frac{x_2}{(1 + r/12)^2} = \frac{2}{(1 + 0.5\%)^2}$$

Present value

- monthly cash flow stream: $(-3, 2, 2)$
- annual interest rate $r = 6\%$

The **present value** of this cash flow stream is

$$\begin{aligned}\text{PV} =& x_0 + \underbrace{\frac{x_1}{1 + r/12}}_{\text{discounted}} + \underbrace{\frac{x_2}{(1 + r/12)^2}}_{\text{discounted}} \\ =& -3 + \frac{2}{1 + 0.5\%} + \frac{2}{(1 + 0.5\%)^2} \approx 0.97\end{aligned}$$

General ideas about present value

- a cash flow stream $(x_0, x_1, \ldots, x_T)$
- will receive cash at m equally spaced periods per year
- interest rate of each period is r/m

The present value of this cash flow stream is

$$\text{PV} = x_0 + \frac{x_1}{1 + r/m} + \frac{x_2}{(1 + r/m)^2} + \ldots + \frac{x_T}{(1 + r/m)^T}$$

Decision according to present value

PV = sum of **discounted** cash flows

Accept a project only if it has a **positive** present value, in other words, only if you can make a profit.

An example of computing present value

- monthly cash flow stream: $(-3, 2, 2)$
- annual interest rate $r = 6\%$

The **present value** of this cash flow stream is

$$\begin{aligned} \text{PV} =& x_0 + \underbrace{\frac{x_1}{1 + r/12}}_{\text{discounted}} + \underbrace{\frac{x_2}{(1 + r/12)^2}}_{\text{discounted}} \\ =& -3 + \frac{2}{1 + 0.5\%} + \frac{2}{(1 + 0.5\%)^2} \approx 0.97 \end{aligned}$$

Example

Suppose that the 'profit periods' have been extended

- initial investment of 3 thousand dollars
- receives 2 thousand dollars in each of the next **36** months
- $x_0 = -3$
- $x_t = 2$, for $t = 1, \ldots, 36$.

The cash flow stream is then

$$(-3, \underbrace{2, 2, \ldots, 2}_{\text{36 months}})$$

Sequence of discounted cash flow

Assume again the annual interest rate is $r = 6\%$. The **sequence** of discounted cash flows is:

$$-3,\ \frac{2}{1+6\%/12},\ \frac{2}{(1+6\%/12)^2},\ \ldots,\ \frac{2}{(1+6\%/12)^{36}}$$

The present value

Sum of the discounted cash flow

The present value

$$\mathrm{PV} = -3 + \frac{2}{1+6\%/12} + \ldots + \frac{2}{(1+6\%/12)^{36}} = ?$$

Sequence

A sequence is an ordered list of numbers

$$T_1, T_2, T_3, T_4, \ldots$$

An example: $2, 5, 8, 11, 14, \ldots$.

- $T_1 = 2$: the first term
- $T_2 = 5$: the second term
- $T_3 = 8$: the third term
- $T_4 = 11$: the fourth term
- $T_5 = 14$: the fifth term

Arithmetic sequence

- It is a sequence where the difference between any pair of successive terms is the same constant.
- The nth term of an arithmetic sequence is

$$T_n = a + (n-1) \cdot d$$

- $a = T_1$ is the first term
- $d = T_{n+1} - T_n$, $n \geq 1$, is the **constant** difference between successive terms

Example ($a = 2$ and $d = 3$)

$T_1 = 2$, $T_2 = T_1 + 3 = 5$, $T_3 = T_2 + 3 = 8, \ldots$
This sequence is $2, 5, 8, 11, 14, \ldots$.

Geometric sequence

- It is a sequence where the ratio between any pair of successive terms is the same constant.
- The nth term is $T_n = a \cdot K^{n-1}, \quad a, K > 0$.
- $a = T_1$ first term of the sequence
- $K = T_{n+1}/T_n$ is the ratio between successive terms

Example ($a = 4$ and $K = 2$)

$T_1 = a = 4$, $T_2 = 2 \cdot T_1 = 8$, $T_3 = 2 \cdot T_2 = 16, \ldots$
This sequence is $4, 8, 16, 32, 64, \ldots$

Finite sum of sequence: A series

Given a sequence $T_1, T_2, \ldots, T_n, \ldots$,
A **series** is a sequence $S_1, S_2, \ldots, S_n, \ldots$, given by

$$S_n = \sum_{i=1}^{n} T_i = T_1 + T_2 + \ldots + T_n$$

which is a finite sum of the sequence T_n.

Example

The series for the sequence $1, 3, 5, 7, 9, \ldots$ is

$$S_1 = 1, \quad S_2 = 1 + 3 = 4, \quad S_3 = 1 + 3 + 5 = 9, \ldots$$

Arithmetic series

The series of the arithmetic sequence

$$T_n = a + (n-1) \cdot d, \quad n = 1, 2, \ldots$$

is given by

$$\begin{aligned} S_n &= a + (a + d) + (a + 2d) + \ldots + (a + (n-1) \cdot d) \\ &= na + \underbrace{(0 + 1 + 2 + \ldots + (n-1))}_{n \text{ terms}} \cdot d \\ &= na + \frac{n}{2} \cdot (0 + (n-1)) \cdot d = na + \frac{n(n-1)}{2} \cdot d \end{aligned}$$

Arithmetic series as finite sum of an arithmetic sequence

$$S_n = na + \frac{n(n-1)}{2} \cdot d, \qquad \text{or } S_n = \frac{n}{2} \cdot (T_1 + T_n).$$

Finite sum of senquential whole numbers

Why $0 + 1 + 2 + \ldots + (n-1) = \frac{n(n-1)}{2}$?

$$\begin{array}{cccccccccc} x & = & 0 & + & 1 & + & \ldots & + & (n-1) \\ x & = & (n-1) & + & (n-2) & + & \ldots & + & 0 \\ \hline 2x & = & (n-1) & + & (n-1) & + & \ldots & + & (n-1) \end{array}$$

Hence,

$$2x = n \cdot (n-1) \quad \Rightarrow \quad x = \frac{n(n-1)}{2}$$

This is simply one method to prove the finite sum formula.

Example

The arithmetic sequence with $a = 2$, $d = 0.5$ is

$$2,\ 2.5,\ 3,\ 3.5,\ 4,\ 4.5,\ 5,\ 5.5, \ldots$$

Determine its series S_n, and compute S_{11}.

$$S_n = n \cdot 2 + \frac{n(n-1)}{2} \cdot 0.5 = 0.25n^2 + 1.75n$$

$$S_{11} = 0.25 \cdot 11^2 + 1.75 \cdot 11 = 49.5$$

Alternative: $T_1 = 2$, $T_{11} = 2 + 10 \cdot 0.5 = 7$

$$S_{11} = \frac{11}{2} \cdot (T_1 + T_{11}) = \frac{11}{2} \cdot 9 = 49.5$$

Geometric Series

The series of the geometric sequence ($a > 0$ and $K > 0$)

$$T_n = aK^{n-1}, \quad n = 1, 2, \ldots$$

is given by

$$\begin{aligned} S_n =& a + aK + aK^2 + \ldots + aK^{n-1} \\ =& a \cdot (1 + K + K^2 + \ldots + K^{n-1}) \\ =& \begin{cases} a \cdot \frac{1-K^n}{1-K} & \text{if } K \neq 1 \\ a \cdot n & \text{if } K = 1 \end{cases} \end{aligned}$$

If $0 < K < 1$, as $n \to \infty$,

$$K^n \to 0, \quad S_n \to a \cdot \frac{1}{1-K} = T_1 + T_2 + \ldots + T_n + \ldots$$

Finite sum of geometric sequence

When $K \neq 1$, show that $1 + K + K^2 + \ldots + K^{n-1} = \frac{1-K^n}{1-K}$.

$$\begin{array}{rcccccccc} x = & 1 & + & K & + & \ldots & + & K^{n-1} & & \\ \hline Kx = & & & K & + & \ldots & + & K^{n-1} & + & K^n \end{array}$$

Hence,

$$(1-K)x = 1 - K^n \quad \Rightarrow \quad x = \frac{1-K^n}{1-K}$$

Example

The geometric sequence with $a = 2$ and $K = \frac{3}{4}$ is

$$2, \frac{3}{2}, \frac{9}{8}, \frac{27}{32}, \frac{81}{128}, \ldots$$

Note that $K \neq 1$, and then we have

$$S_n = a \cdot \frac{1-K^n}{1-K} = 2 \cdot \frac{1-(3/4)^n}{1-3/4} = 8 - 8 \cdot \left(\frac{3}{4}\right)^n$$

As $n \to \infty$, we have $S_n \to 8 - 8 \cdot 0 = 8$.

Annuity

An annuity is a contract that pays the holder money periodically, that is, with the cash flow stream:

$$(0, \underbrace{A, A, \ldots, A}_{T \text{ periods}})$$

for some constant $A > 0$.

In the last example, if the initial investment is 0, it is an annuity

$$(0, \underbrace{2, 2, \ldots, 2}_{\text{T=36 months}}).$$

PV of annuity $(0, A, A, \ldots, A)$

Suppose the **period interest** is r_m. The sequence of discounted cash flow is

$$0, \underbrace{\frac{A}{1+r_m}, \frac{A}{(1+r_m)^2}, \ldots, \frac{A}{(1+r_m)^T}}_{\text{Geometric Sequence}}$$

$$\mathrm{PV} = \underbrace{\frac{A}{1+r_m} + \frac{A}{(1+r_m)^2} + \ldots + \frac{A}{(1+r_m)^T}}_{\text{Geometric Series}}$$

$$= \frac{A}{1+r_m} \cdot \frac{1 - \left(\frac{1}{1+r_m}\right)^T}{1 - \frac{1}{1+r_m}} = \frac{A}{r_m} \cdot \left(1 - \left(\frac{1}{1+r_m}\right)^T\right)$$

Perpetual annuity ($T = \infty$)

This type of annuity pays a fixed amount A periodically forever. As the $T \to \infty$, we have

$$\frac{A}{r_m} \cdot \left(1 - \left(\frac{1}{1+r_m}\right)^T\right) \to \frac{A}{r_m} \cdot (1-0) = \frac{A}{r_m}$$

which is the net PV of the perpetual annuity $(0, A, A, \ldots)$.

Our example

In the last example, if initial investment is 0, it is annuity

$$(0, \underbrace{2, 2, \ldots, 2}_{\text{T=36 months}}).$$

Assume the annual interest rate is $r = 6\%$ (r_m=6%/12=0.5%)

$$\text{PV} = \frac{A}{r_m}\left(1 - \left(\frac{1}{1+r_m}\right)^T\right) = \frac{2}{0.5\%}\left(1 - \left(\frac{1}{1+6\%/12}\right)^{36}\right) \approx 65.74$$

Loan (debt repayments)

A loan is said to be amortized if

- both the principal L (amount borrowed)
- and the interest with nominal rate r
- are repaid by equal payments A_0
- over t years (m times payments in a year)
- at $n = m \cdot t$ equally-spaced intervals

If you buy a loan contract (lend out money), your cash flow stream is given by

$$(-L, A_0, A_0, \ldots, A_0)$$

We say the payment A_0 is '**fair**' if the present value of this cash flow stream is **zero**.

Fair repayments

The cash flow stream

$$(-L, \quad \underbrace{A_0, A_0, \ldots, A_0}_{\text{compare with annuity}} \quad)$$

has present value

$$\mathrm{PV} = -L + \frac{A}{r/m}\left(1 - \left(\frac{1}{1 + r/m}\right)^n\right) = 0$$

Solve this equation to get

$$A_0 = L \cdot \frac{r/m}{1 - (1 + r/m)^{-n}}$$

Example

You want to buy a house in Melbourne. For this you need to get a loan of $200,000$ dollars to be repaid over 25 years at interest rate 4.5%. Calculate the monthly repayment amount.

- $L = 200000$, $r = 4.5\%$, $m = 12$, $n = 12 \times 25 = 300$

Hence, the monthly repayment amount is

$$\begin{aligned} A_0 =& L \cdot \frac{r/m}{1-(1+r/m)^{-n}} \\ =& 200000 \cdot \frac{0.045/12}{1-(1+0.045/12)^{-300}} \approx 1111.66 \end{aligned}$$

Flat rate loan

Banks could provide the so-called **flat rate loan** for which the repayment amount can be easier to be calculated.

- use the so called *simple interest rate* instead of the effective interest rate.

Simple Interest Rate

The t-year (possibly $t < 1$) simple interest rate is rt.

The repayment amount A_0 for flat-rate loan satisfies

$$nA_0 = (1 + r_{\text{flat}} \cdot t)\, L, \Rightarrow A_0 = L \cdot \frac{1}{n} (1 + r_{\text{flat}} \cdot t)$$

Recall from the previous example:

- $L = 200000$, $r = 4.5\%$, $m = 12$, $t = 25$, $n = 300$

and the monthly repayment amount is $A_0 \approx 1111.66$

Determine the interest rate for a flat-rate loan with the same principal and paid-over periods that requires the same monthly repayment amount.

Solve $A_0 = L \cdot \frac{1}{n}\left(1 + r_{\text{flat}} \cdot t\right)$ to get

$$r_{\text{flat}} = \frac{1}{t}\left(\frac{nA_0}{L} - 1\right) \approx \frac{1}{25}\left(\frac{300 \times 1111.66}{200000} - 1\right)$$
$$\approx 2.67\%$$

So far, we have discussed:

- present value of a cash flow stream (for an investment project)
- a single project evaluation: take on the project only if it has positive present value
- present value of an (perpetual) annuity
- how to use present value technique to determine the repayment amounts for loans

Next, we will discuss how to compare investment projects using present value techniques.

Comparison: Investment projects

Present value criterion

If you can only take on one project, take the one with maximal (positive) present value (i.e. the one you can earn most).

Projects with (annual) cash flow streams:

- $(-1, 2)$
- $(-1, 0, 3)$

The interest rate is 10%. The present values:

- $PV_1 = -1 + \frac{2}{1.1} \approx 0.82$
- $PV_2 = -1 + \frac{3}{1.1^2} \approx 1.48$

Hence to choose the second project.

When interest rate r is unknown

$$\mathrm{PV}_1 = -1 + \tfrac{2}{1+r}, \mathrm{PV}_2 = -1 + \tfrac{3}{(1+r)^2}$$

$$\begin{aligned} \mathrm{PV}_1 > \mathrm{PV}_2 \Leftrightarrow & -1 + \frac{2}{1+r} > -1 + \frac{3}{(1+r)^2} \\ \Leftrightarrow & 1 + r > \frac{3}{2} \Leftrightarrow r > \frac{1}{2} \end{aligned}$$

- $r \in \left(\frac{1}{2}, 1\right)$: $\mathrm{PV}_1 > \mathrm{PV}_2 \Rightarrow$ choose the first project
- $r \in \left(0, \frac{1}{2}\right)$: $\mathrm{PV}_1 < \mathrm{PV}_2 \Rightarrow$ choose the second project

Which project to take?

Internal rate of return

The **internal rate of return** r of a cash flow stream $(x_0, x_1, \ldots, x_T)$ is the solution of the equation

$$\text{PV} = x_0 + \frac{x_1}{1+r} + \ldots + \frac{x_T}{(1+r)^T} = 0$$

- $(-1, 2)$: Solve $-1 + \frac{2}{1+r} = 0$ to obtain $r = 1 = 100\%$
- $(-1, 0, 3)$: Solve $-1 + \frac{3}{(1+r)^2} = 0$ to obtain $r = \sqrt{3} - 1 \approx 73.21\%$

Alternative criteria: Internal rate of return criterion

Choose the project with maximal internal rate of return.

- $(-1, 2)$: $IRR_1 = 100\%$
- $(-1, 0, 3)$: $IRR_2 \approx 73.21\%$

Hence according to the internal rate of return criteria, we should take on the first project.

PV or IRR?

- Present value: how much you can earn
- IRR: how fast you can earn

Advice

Use present value criteria whenever possible

- The IRR criteria can be misleading: for example, a project may have positive internal rate of return, but negative present value.
- We may know a 'reasonable' range of r in real life. In the above example, a practical discount rate r is easily to be less than 50%, then we should take on the 2nd project.