ETF2700/ETF5970 Mathematics for Business

Lecture 8

Monash Business School, Monash University, Australia

Outline

Weeks 4-7:

- Derivative: Single variable
- Partial derivative: multiple variables
- Optimisation: stationary points
- Optimisation with constraint: Lagrange method

This week:

- Indefinite and definite Integration
- Differential Equation

Integration as anti-differentiation

- Differentiation of $F(x)$ gives its derivative $F'(x)$, which is also a function of x.
- We can treat $F'(x)$ as giving the slope of the tangent line at the point x on the graph of $y = F(x)$.
- For a given function $f(x)$, integration is to find a function $F(x)$, such that $F'(x) = f(x)$.
- As the derivative of any constant is zero, we can only determine $F(x)$ up to an additive constant.
- Integration as taking anti-derivative of $f(x)$ is $\int f(x)dx$

Integration as anti-derivative

- The outcome of integration is not just one function, but **a group** of functions, all having the same derivative $f(x)$. As such it is called an **indefinite integral** of $f(x)$.
- To integrate $f(x) = x$ w.r.t. x, we know that $(x^2)' = 2x$. So, $(\frac{1}{2}x^2)' = x$. This integration is written as $\int xdx = \frac{1}{2}x^2 + c$

How to solve an indefinite integral

As the derivative of x^n is nx^{n-1}, therefore,

$$\left(n^{-1}x^n\right)' = x^{n-1}$$

Thus, for integer $n \neq -1$, we have

$$\int x^{n-1}dx = n^{-1}x^n + C$$
$$\int x^n dx = (n+1)^{-1}x^{n+1} + C$$

How to integrate x^{-1}? Note that $(\ln x)' = x^{-1}$, thus

$$\int x^{-1}dx = \ln x + C, \quad \text{for } x > 0.$$

Primitive function

- In general, we can write the integral of $f(x)$ as

$$\int f(x)dx = F(x) + C$$

  $F(x)$ is called the primitive function of $f(x)$.

- Sometimes we know the primitive function of $f(x)$ exists, but we don't know its analytical form. For example,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}.$$

  Its primitive function exists, but its analytical form is unknown

Indefinite integral: Examples

$f(x)$	$F(x)$	$\int f(x)dx$
k (constant)	kx	$kx + C$
x	$\frac{1}{2}x^2$	$\frac{1}{2}x^2 + C$
x^n $(n \neq -1)$	$(n+1)^{-1}x^{n+1}$	$(n+1)^{-1}x^{n+1} + C$
e^x	e^x	$e^x + C$
a^x $(a \neq 1)$	$\frac{a^x}{\ln(a)}$	$\frac{a^x}{\ln(a)} + C$
x^{-1} $(x > 0)$	$\ln(x)$	$\ln(x) + C$
x^{-1} $(x \neq 0)$	$\ln(\lvert x \rvert)$	$\ln(\lvert x \rvert) + C$
$\sin(x)$	$-\cos(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x)$	$\sin(x) + C$

- We can verify that for each $f(x)$ given above, $F'(x) = f(x)$
- In general, it is difficult to find $F(x)$.
- Sometimes it is impossible to derive an analytical form of $F(x)$.

Sum rule: Indefinite integral

If $F_1'(x) = f_1(x)$ and $F_2'(x) = f_2(x)$, then

$$\int (f_1(x) + f_2(x))dx = \int f_1(x)dx + \int f_2(x)dx = F_1(x) + F_2(x) + C$$

We can prove it by the sum rule of derivatives:

$$F_1(x) + F_2(x) \text{ is a primitive of } f_1(x) + f_2(x)$$

Subtraction rule: Indefinite integral

$$\int (f_1(x) - f_2(x))dx = \int f_1(x)dx - \int f_2(x)dx = F_1(x) - F_2(x) + C$$

with $F_1'(x) = f_1(x)$ and $F_2'(x) = f_2(x)$

We can prove by subtraction rule of derivatives:

$$F_1(x) - F_2(x) \text{ is a primitive of } f_1(x) - f_2(x)$$

Product with constant

For any constant $K \in (-\infty, \infty)$, we have

$$\int K \cdot f(x)dx = K \cdot \int f(x)dx = K \cdot F(x) + C,$$

We can prove it by the product rule for differentiation

$$K \cdot F(x) \text{ is a primitive of } K \cdot f(x)$$

For any $F(x)$ such that $F'(x) = f(x)$, we have

$$\int f(x)dx = F(x) + C$$

Differentiation		Integration
Sum Rule	$\Leftrightarrow$	Sum Rule
Subtraction Rule	$\Leftrightarrow$	Subtraction Rule
Product Rule (Constant)	$\Leftrightarrow$	Product Rule (Constant)
Product Rule (General)	$\Leftrightarrow$	?
Chain Rule	$\Leftrightarrow$	?

Integration by parts

Differentiation		Integration
Sum Rule	$\Leftrightarrow$	Sum Rule
Subtraction Rule	$\Leftrightarrow$	Subtraction Rule
Product Rule (Constant)	$\Leftrightarrow$	Product Rule (Constant)
Product Rule (General)	$\Leftrightarrow$	Integration by Parts
Chain Rule	$\Leftrightarrow$	?

Let $f(x) = u(x) \cdot v(x)$. By the product rule, we have

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

and therefore

$$u(x)v'(x) = f'(x) - u'(x)v(x) =: f'(x) - g'(x)$$

$$\int u(x)v'(x)dx = f(x) - g(x) + C = f(x) - (g(x) + \widetilde{C})$$

where C and $\widetilde{C} = -C$ are arbitrary constants.

$$\int u(x)v'(x)dx = u(x) \cdot v(x) - \int u'(x)v(x)dx$$

Example

Use integration by parts to evaluate $\int xe^x dx$.

Note that $xe^x = u(x) \cdot v'(x)$, where

$$u(x) = x, \quad v(x) = e^x.$$

and $u'(x) = 1$.

$$\begin{aligned}\int \underbrace{x}_{u(x)} \cdot \underbrace{e^x}_{v'(x)} dx = & \underbrace{x}_{u(x)} \cdot \underbrace{e^x}_{v(x)} - \int \underbrace{1}_{u'(x)} \cdot \underbrace{e^x}_{v(x)} dx \\ = & x \cdot e^x - (e^x + \widetilde{C}) = xe^x - e^x + C\end{aligned}$$

where $C = -\widetilde{C}$ is an arbitrary constant.

Integration by Substitution

Differentiation	Integration
Sum Rule	Sum Rule
Subtraction Rule	Subtraction Rule
Product Rule (Constant)	Product Rule (Constant)
Product Rule (General)	Integration by Parts
Chain Rule	Integration by Substitution

Let $g(x) = F(\varphi(x))$. By the chain rule we have

$$g'(x) = f(\varphi(x))\varphi'(x)$$

By definition, we have

$$\int f(\varphi(x))\varphi'(x)dx = g(x) + C = F(\varphi(x)) + C$$

Integration by Substitution

$$\int f(\varphi(x))\varphi'(x)dx = F(\varphi(x)) + C$$

- dx = infinitely small changes in x
- $d\varphi(x)$ = resulting infinitely small changes in $\varphi(x)$
- $\varphi'(x) = \frac{d\varphi(x)}{dx} \quad \Rightarrow \quad d\varphi(x) = \varphi'(x)dx$
- Substitute $d\varphi(x) = \varphi'(x)dx$

$$\int f(\varphi(x))\varphi'(x)dx = \int f(\varphi(x))d\varphi(x)$$

- Substitute $z = \varphi(x)$

$$\int f(z)dz = F(z) + C = F(\varphi(x)) + C$$

Example

Use integration by substitution to evaluate $\int 2xe^{x^2}\,dx$.

Write $2xe^{x^2} = e^{x^2} \cdot (2x) = f(\varphi(x)) \cdot \varphi'(x)$ with

$$\varphi(x) = x^2, \quad f(x) = e^x.$$

A primitive of $f(x) = e^x$ is $F(x) = e^x$.

$$\int 2xe^{x^2}\,dx = F(\varphi(x)) + C = e^{x^2} + C$$

where C is an arbitrary constant.

Definite Integrals

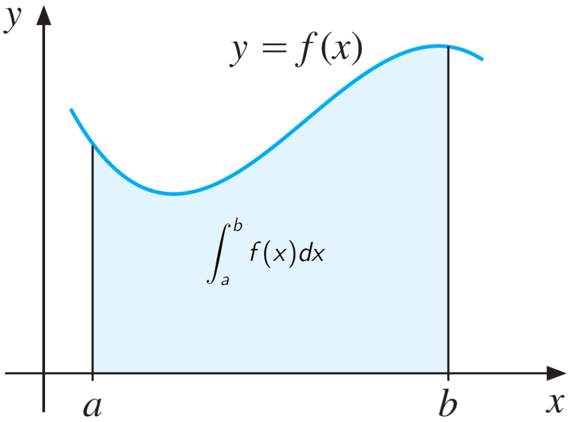
- Let $f(x)$ to be a continuous function on $[a, b]$
- $F'(x) = f(x), x \in (a, b)$: $F(x)$ is the primitive function on (a, b)

The **definite integral** of $f(x)$ over $[a, b]$ is

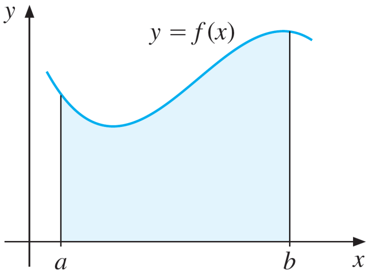
$$\int_a^b f(x)dx = F(b) - F(a)$$

- It does not depend on the choice of F
- Result is just a (real) number, not a function
- $F(b) - F(a)$ can be expressed as $[F(x)]_a^b$ or $F(x)\Big|_{x=a}^{x=b}$ or $[F(x)]_{x=a}^{x=b}$

Definite integral as an area



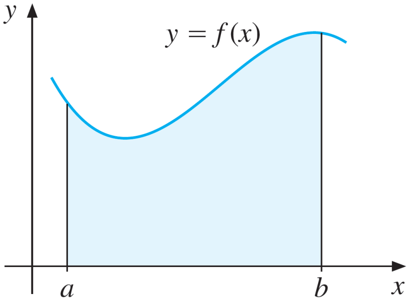
Definite integral as an area



Let $\Delta t = \frac{b-a}{M}$, $t_0 = a$, and $t_i = a + i \cdot \Delta t$, for $i = 1, \ldots, M$

$$\begin{aligned}
\text{Blue Area} \approx & f(t_1)\cdot(t_1 - t_0) + f(t_2)\cdot(t_2 - t_1) \\
& + \ldots + f(t_M)\cdot(t_M - t_{M-1}) \\
= & \sum_{a<t_i\leq b} f(t_i)\cdot(t_i - t_{i-1}) \\
& \xrightarrow{\Delta t \downarrow 0 \quad (M\to\infty)} \int_a^b f(t)dt
\end{aligned}$$

Definite Integral as summation



$$\sum_{a<t_i\leq b} f(t_i)\cdot(\underbrace{t_i - t_{i-1}}_{\Delta t}) \xrightarrow{\Delta t\downarrow 0} \int_a^b f(t)dt = \text{Blue Area}$$

Sum Rule

$$\int_a^b (f_1(t)+f_2(t))dt = \int_a^b f_1(t)dt + \int_a^b f_2(t)dt$$

Subtraction rule

$$\int_a^b (f_1(t)-f_2(t))dt = \int_a^b f_1(t)dt - \int_a^b f_2(t)dt$$

Multiplication by a constant

$$\int_a^b K \cdot f_1(t)dt = K \cdot \int_a^b f_1(t)dt, \quad K \in (-\infty, \infty)$$

Total Cost Function

Consider a firm that produces x units of jackets.

- Its fixed cost is \$240
- Its variable cost for each unit is \$60
- Total cost (function) in dollars:

$$\underbrace{60 + \ldots + 60}_{x \text{ times}} + \underbrace{240}_{\text{Fixed Cost}} = 60x + 240$$

Functional variable cost

- Assume the fixed cost is \$240
- Cost of one additional unit at production x is:

$$f(x) = 60 + x, \quad x = 0, 1, \ldots$$

- Let $F(x)$ denote the total cost. At the production of x units, the cost of an additional unit is:

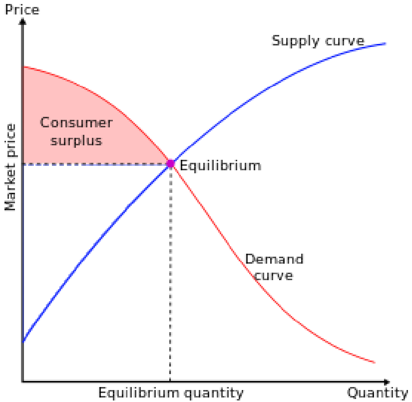
$$F(x+1) - F(x) = f(x) = 60 + x$$

Our Example

Beginning with $f(x) = x + 60$ with a primitive $G(x) = \frac{1}{2}x^2 + 60x$, we have the total cost function as

$$\begin{aligned}
F(x) &= \underbrace{F(x) - F(0)}_{\text{Variable Cost}} + \underbrace{F(0)}_{\text{Fixed Cost}} \\
&= \int_0^x f(t)dt + F(0) \\
&= (G(x) - G(0)) + F(0) \\
&= \left(\frac{1}{2}x^2 + 60x - 0\right) + 240 \\
&= \frac{1}{2}x^2 + 60x + 240
\end{aligned}$$

Consumer Surplus



- market demand Q
- market price

$$P = 100 - Q$$

- in equilibrium:

$$P_0 = 40,\ Q_0 = 60$$

$$\text{CS} = \int_0^{Q_0} (P(Q) - P_0)dQ = \underbrace{\int_0^{Q_0} P(Q)dQ}_{\text{Willing to Pay}} - \underbrace{P_0 \cdot Q_0}_{\text{Actual Pay}}$$

Calculate CS

Recall $P_0 = 40$, $Q_0 = 60$ and

$$P(Q) = 100 - Q$$

which has a primitive $F(Q) = 100Q - \frac{1}{2}Q^2$

$$\begin{aligned} \text{CS} &= \int_0^{Q_0} P(Q)dQ - P_0 \cdot Q_0 \\ &= (F(Q_0) - F(0)) - P_0 \cdot Q_0 \\ &= (F(60) - F(0)) - 40 \cdot 60 \\ &= 4200 - 2400 = 1800. \end{aligned}$$

Integration by Parts: definite integral

Let $f(x) = u(x) \cdot v(x)$. We know

$$u(x)v'(x) = f'(x) - u'(x)v(x)$$

$$\int_a^b u(x)v'(x)dx = \int_a^b f'(x)dx - \int_a^b u'(x)v(x)dx$$

$$\int_a^b u(x)v'(x)dx = (f(b) - f(a)) - \int_a^b u'(x)v(x)dx$$

$$\int_a^b u(x)v'(x)dx = u(x) \cdot v(x)\Big|_{x=a}^{x=b} - \int_a^b u'(x)v(x)dx$$

Example

Use integration by parts to evaluate $\int_0^1 xe^x dx$.

Note that $xe^x = u(x) \cdot v'(x)$, where

$$u(x) = x, \quad v(x) = e^x.$$

and $u'(x) = 1$.

$$\int_0^1 \underbrace{x}_{u(x)} \cdot \underbrace{e^x}_{v'(x)} dx = \underbrace{x}_{u(x)} \cdot \underbrace{e^x}_{v(x)} \Big|_{x=0}^{x=1} - \int_0^1 \underbrace{1}_{u'(x)} \cdot \underbrace{e^x}_{v(x)} dx$$

$$= (1 \cdot e^1 - 0 \cdot e^0) - (e^1 - e^0) = 1$$

Integration by Substitution

Let $g(x) = F(\varphi(x))$. By chain rule we have

$$g'(x) = f(\varphi(x))\varphi'(x)$$

By definition, we have

$$\begin{aligned}\int_a^b f(\varphi(x))\varphi'(x)dx =& F(\varphi(b)) - F(\varphi(a)) \\ =& \int_{\varphi(a)}^{\varphi(b)} f(x)dx\end{aligned}$$

Example

Use integration by substitution to evaluate $\int_0^2 2xe^{x^2}\,dx$.

Write $2xe^{x^2} = e^{x^2} \cdot (2x) = f(\varphi(x)) \cdot \varphi'(x)$ with

$$\varphi(x) = x^2, \quad f(x) = e^x.$$

A primitive of $f(x) = e^x$ is $F(x) = e^x$.

$$\begin{aligned}\int_0^2 2xe^{x^2}\,dx &= \int_{\varphi(0)}^{\varphi(2)} f(x)dx \\ &= \int_0^4 e^x dx = e^x\Big|_{x=0}^{x=4} = e^4 - 1.\end{aligned}$$

Function as an unknown argument

- We know $y = F(x)$, but the analytical form of F is unknown.
- the derivative of $F(x)$ is $f(x)$, which is *known*.

The primitive function of $f(x)$ is the solution of the following equation:

$$F'(x) = f(x), \text{or often written as } \frac{dy}{dx} = f(x)$$

where $y = F(x)$.

- The unknown is a **function**, not a variable
- It involves the function $F(x)$ and its derivative
- Such an equation is called **differential equation**

First-order linear differential equation

A first-order linear differential equation in $F(x)$ is

$$F'(x) = a(x) \cdot F(x) + b(x)$$

where $a(x)$ and $b(x)$ are **known** functions.
Let $y = F(x)$, we may write the equation in y as

$$\frac{dy}{dx} = a(x) \cdot y + b(x).$$

- When $a(x) \equiv 0$: the solution set is $\int b(x)dx$

A Special Case

Let $a(x) = k$ for all x and $b(x) = 0$.

$$\frac{dy}{dx} = ky, \quad k \text{ is a known constant}$$

Can we solve y? and how can we solve it?

A Heuristic way: Textbook

WARNING: This procedure is NOT correct!

1) Divide y on both sides: $\frac{1}{y}dy = kdx$
2) Integrate both sides: $\ln(|y|) + C_1 = kx + C_2$
3) $|y| = \widetilde{A}e^{kx}$ with $\widetilde{A} = e^{C_2 - C_1} > 0$
4) $y = Ae^{kx}$ with $A \neq 0$

However, this helps us to **guess** the solution

A correct way

1) Guess the solution: $y = Ae^{kx}$ for some A
2) Plug Ae^{kx} into the equation $\frac{dy}{dx} = ky$:

$$\text{LHS}=Ae^{kx} \cdot k = Ake^{kx},$$
$$\text{RHS}=k \cdot Ae^{kx}=\text{LHS}$$

So A can be any real number (including 0).
3) We can prove that these are the only possible solution (homework).
Hint: We may assume the solution to be $y = \lambda(x) \cdot e^{kx}$, and show that $\lambda'(x) \equiv 0$. Thus $\lambda(x)$ can only be a constant.

General solution

$y = Ae^{kx}$, where A is an arbitrary constant.

General linear differential equations

- A general first-order linear differential equation is

$$\frac{dy}{dx} = ky + b, \text{ where } k \text{ and } b \text{ are known constants}$$

  The equation is **homogeneous** if $b = 0$; otherwise it is **inhomogeneous** or **heterogeneous**.
- The general solution of a heterogeneous differential equation can be obtained based on the general solution of the corresponding homogeneous equation (Section 12.3 of [MP]).

General solution

The general solution to $\frac{dy}{dx} = ky + b$ is

$$y = A \cdot e^{kx} - \frac{b}{k},$$

with A being a constant to be determined by an initial condition, for $k \neq 0$.

When $k = 0$, the general first-order linear differential equation becomes

$$\frac{dy}{dx} = b, \quad \text{where } b \text{ is a known constant}$$

Its solution is $y = bx + c$ with c to be determined by an initial condition.

Initial condition

- Initial condition: If we know $y(x_0) = y_0$ for some known (x_0, y_0), we can plug in (x_0, y_0) to solve A (and c in the case of $k = 0$).
- For more complex equations, in general, we do not always know (how to guess) the solutions.

Example

We know the function $y = y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = 5y$$

and $y(0) = 3$. Determine the function y.

The solution of the differential equation is

$$y = Ae^{5x} \quad \text{for any constant } A \in (-\infty, \infty)$$

Then $3 = y(0) = A \cdot e^{5 \cdot 0} = A$. Hence,

$$y = 3e^{5x}.$$

Another example

A model for the population $y(t)$, in millions, of a country at time t shows that the rate of change of the population is given by

$$\frac{dy}{dt} = -0.05y + 4.5$$

The population at time $t = 0$ is 100 million. Solve the function $y(t)$.
The general solution is

$$y(t) = A \cdot e^{-0.05t} - \frac{4.5}{-0.05}.$$

Thus, we have $y(t) = A \cdot e^{-0.05t} + 90$.
Plugging in the initial condition $y(0) = 100$, we obtain that $A = 10$. Thus $y(t) = 10 \cdot e^{-0.05t} + 90$.