ETF2700/ETF5970 Mathematics for Business

Lecture 11

Monash Business School, Monash University, Australia

Outline

Last two weeks:

- Basic theory of interest
- Cash flow stream: sequences and series
- Present value techniques and applications
- Depreciation, inflation, and real growth
- Difference equations

This week:

- More about difference equations

Student Evaluation of Teaching and Units, SETU

- All students are encouraged to complete SETU Survey on Moodle
- Your feedback is highly appreciated

Review: First-order linear difference equation

A first-order linear difference equation

$$Y_{t+1} = aY_t + b$$

- a, b and Y_0 are given
- First-order: Y_{t+1} is fully determined by the 1-period lagged value Y_t
- Linear: Relationship between Y_{t+1} and Y_t is linear
- $a = 1$: An arithmetic sequence so $Y_t = Y_0 + t \cdot b$

Review: Solve $Y_{t+1} = aY_t + b$, for $a \neq 1$

1) Rewrite difference equation

$$\underbrace{Y_{t+1} - \frac{b}{1-a}}_{\widetilde{Y}_{t+1}} = a \cdot \underbrace{\left(Y_t - \frac{b}{1-a}\right)}_{\widetilde{Y}_t}$$

2) $\widetilde{Y}_t = Y_t - \frac{b}{1-a}$ is a geometric sequence

$$\widetilde{Y}_t = a^t \widetilde{Y}_0 = a^t \left(Y_0 - \frac{b}{1-a}\right)$$

3) Write

$$Y_t - \frac{b}{1-a} = a^t \left(Y_0 - \frac{b}{1-a}\right)$$

$$Y_t = \frac{b}{1-a} + a^t \left(Y_0 - \frac{b}{1-a}\right)$$

Review: Solution to $Y_{t+1} = aY_t + b$

When $a \neq 1$,

$$Y_t = \widetilde{Y}_t + \frac{b}{1-a} = a^t\left(Y_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$$

Example

In the last week's example, you have an initial savings of \$1000 in your bank account. In each month, you spend 80% of your last month's savings; and you receive an income of \$5000.
To summarise, we have $a = 0.2$, $b = 5000$ and $Y_0 = 1000$. Thus

$$\begin{aligned} Y_t =& 0.2^t\left(1000 - \frac{5000}{1-0.2}\right) + \frac{5000}{1-0.2} \\ =& 0.2^t(-5250) + 6250 \end{aligned}$$

Review: An equilibrium state or stationary state

A point Y^* is an equilibrium/stationary state if

$$Y_M = Y^* \quad \Rightarrow Y_t = Y^* \text{ for all } t \geq M$$

An equilibrium/stationary state Y^* is a solution of the equation

$$y = ay + b$$

because the subscript t can removed.

- When $a \neq 1$, there is only one equilibrium state $Y^* = \frac{b}{1-a}$
- When $a = 1$ but $b \neq 0$: There is no equilibrium state
- When $a = 1$ but $b = 0$: Any value is an equilibrium $Y^* \in (-\infty, \infty)$, because the sequence Y_t will never change

Review: Stability

- The sequence $Y_t = aY_{t-1} + b$ converges

$$Y_t \to Y^* = \frac{b}{1-a}$$

  for all initial value $Y_0 \in \mathbb{R}$, if and only if $|a| < 1$.
- In this case, we say the difference equation is **globally asymptotically stable**, or sometimes just **stable**.

General situations

- $a > 1$: Y_t is divergent if $Y_0 \neq \frac{b}{1-a}$
- $0 < a < 1$: Y_t achieves uniform convergence
- $-1 < a < 0$: Y_t achieves convergence with oscillation
- $a < -1$: Y_t is divergent with oscillation if $Y_0 \neq \frac{b}{1-a}$

An Example: National Income

Consider a simple model with structural equations:

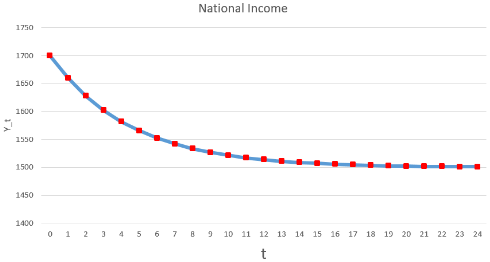
$$Y_t = C_t + I_t, \quad C_t = 0.8Y_{t-1} + 100, \quad I_t = 200$$

with $Y_0 = 1700$.

- Y_t is national income at time t
- C_t is consumption at time t
- I_t is the investment at time t

Time Path

$Y_t = (0.8Y_{t-1} + 100) + 200 = 0.8Y_{t-1} + 300$



Difference Equation: $Y_t = 0.8Y_{t-1} + 300$ with $Y_0 = 1700$

1) Rewrite the difference equation

$$Y_t - \frac{300}{1-0.8} = 0.8 \cdot \left(Y_{t-1} - \frac{300}{1-0.8} \right)$$

2) $\widetilde{Y}_t = Y_t - \frac{300}{1-0.8}$ is a geometric sequence

$$\widetilde{Y}_t = 0.8^t \widetilde{Y}_0 = 0.8^t \left(1700 - \frac{300}{1-0.8} \right) = 200 \cdot 0.8^t$$

3) Write

$$Y_t = \widetilde{Y}_t + \frac{300}{1-0.8} = 200 \cdot 0.8^t + 1500$$

Equilibrium State and Stability

Starting with the difference equation $Y_t = 0.8Y_{t-1} + 300$ with $Y_0 = 1700$, we have now:

$$Y_t = \widetilde{Y}_t + \frac{300}{1 - 0.8} = 200 \cdot 0.8^t + 1500$$

- the **equilibrium state** is $y^* = \frac{300}{1-0.8} = 1500$
- the difference equation is **global asymptotically stable** since the corresponding $a = 0.8 \in (-1, 1)$
- $Y_t \to 1500$, as $t \to \infty$, regardless the value of Y_0

Relationship with differential equation

Consider the differential equation

$$\frac{dy}{dt} = ky$$

Let $y = y(t)$, and according to the first principle, we have

$$\frac{dy}{dt} = y'(t) \approx \frac{y(t+\Delta) - y(t)}{\Delta}$$

Hence, we have

$$\frac{y(t+\Delta) - y(t)}{\Delta} \approx ky(t),$$

$$y(t+\Delta) \approx (1+\Delta k) \cdot y(t)$$

Approximation by difference equation

Recall that

$$y(t+\Delta) \approx (1+\Delta k) \cdot y(t)$$

Define $Y_{t+1} = y(\Delta + t)$ and $Y_t = y(t)$, then

$$Y_{t+1} \approx (1+\Delta k) Y_t, \Rightarrow$$

$$Y_t \approx (1+\Delta k)^t Y_0 = (1+\Delta k)^t y(0)$$

'is' a geometric sequence.

$$y(t) = Y_{t/\Delta} \approx (1+\Delta k)^{t/\Delta} \cdot y(0)$$

Revisit the differential equations

$$y(t) \approx (1+\Delta k)^{t/\Delta} \cdot y(0) = \left((1+\Delta k)^{\frac{1}{\Delta k}}\right)^{kt} y(0).$$

As $\Delta \to 0$, $\Delta k \to 0$ and

$$(1+\Delta k)^{\frac{1}{\Delta k}} \to e$$

so

$$y_t = e^{kt} y(0) = Ae^{kt}$$

where $A = y(0)$ is an arbitrage constant.

A more complicated example

1) Consider a simple model with structural equations

$$Y_t = C_t + I_t, \quad C_t = 0.8Y_{t-1} + 100, \quad I_t = 0.25(C_t - C_{t-1})$$

with $Y_0 = 1700$ and $C_0 = 1000$.

2) To compute Y_1, we have the observations for $t = 1$:

$$\begin{aligned}
C_1 &= 0.8Y_0 + 100 = 1460 \\
I_1 &= 0.25 \cdot (C_1 - C_0) = 0.25(1460 - 1000) = 115 \\
Y_1 &= C_1 + I_1 = 1460 + 115 = 1575
\end{aligned}$$

Recurrence relationship

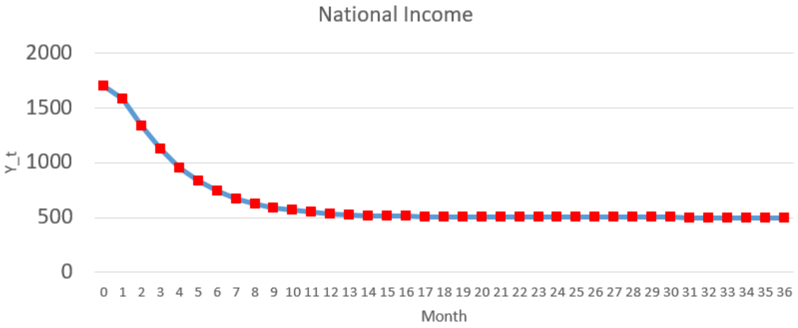
$$\begin{aligned}
I_t =& 0.25(C_t - C_{t-1}) \\
=& 0.25\{(0.8Y_{t-1} + 100) - (0.8Y_{t-2} + 100)\} \\
=& 0.2Y_{t-1} - 0.2Y_{t-2}
\end{aligned}$$

Recurrence relationship

$$\begin{aligned} Y_t &= (0.8Y_{t-1} + 100) + (0.2Y_{t-1} - 0.2Y_{t-2}) \\ &= Y_{t-1} - 0.2Y_{t-2} + 100, \end{aligned}$$

with $Y_0 = 1700$ and $Y_1 = 1575$, for $t = 2, 3, \ldots$

Time path for National Income



Is there an equilibrium state for Y_t?

- If Y_t has an equilibrium state Y_*, then it can be worked out by solving the following equation:

$$Y_* = Y_* - 0.2Y_* + 100$$

- It turns out to be that $Y_* = 500$.
- Do we have any method to justify the equilibrium state?

Solve the difference equation

$$Y_t = Y_{t-1} - 0.2Y_{t-2} + 100, \quad t = 2, 3, \dots$$

with $Y_0 = 1700$ and $Y_1 = 1575$.

Step 1: Express Y_t in the form of a difference equation:

$$Y_t - C = (Y_{t-1} - C) - 0.2\,(Y_{t-2} - C)$$

where we want to know the value C. Rewrite this equation as

$$Y_t = Y_{t-1} - 0.2Y_{t-2} + 0.2C$$

Thus, we obtain that $0.2C$=100 and then C=500.

Transform Y_t to $\widetilde{Y}_t$

Step 2: Let $\widetilde{Y}_t = Y_t - 500$. We have

$$\widetilde{Y}_t = \widetilde{Y}_{t-1} - 0.2\widetilde{Y}_{t-2}$$

Can we solve $\widetilde{Y}_t$?

Rewrite the difference equation as

$$\widetilde{Y}_t - a\widetilde{Y}_{t-1} = K\left(\widetilde{Y}_{t-1} - a\widetilde{Y}_{t-2}\right)$$

What value is a and what value is K?

$$\widetilde{Y}_t = (a+K)\widetilde{Y}_{t-1} - aK\widetilde{Y}_{t-2}$$

Matching it with Step 1, we have

$$a + K = 1 \quad \text{and} \quad a \cdot K = 0.2$$

Work out the values of a and K

$$a + K = 1, \quad a \cdot K = 0.2$$

Actually a and K are the two (different) roots of the quadratic equation:

$$x^2 - x + 0.2 = 0$$

Use the abc-formula to work out the two roots as

$$a = \frac{5 - \sqrt{5}}{10} \approx 0.2764 \text{ and } K = \frac{5 + \sqrt{5}}{10} \approx 0.7236$$

From Y_t to $\widetilde{Y}_t$ and then to $\overline{Y}_t$

Step 3: Let $\overline{Y}_t = \widetilde{Y}_t - a\widetilde{Y}_{t-1}$, then

$$\overline{Y}_t = K\overline{Y}_{t-1}, \quad \Rightarrow \quad \overline{Y}_t = K^{t-1}\overline{Y}_1$$

where

$$\begin{aligned}\overline{Y}_1 =& \widetilde{Y}_1 - a\widetilde{Y}_0 \\ =& (Y_1 - 500) - a(Y_0 - 500) \\ =& 1075 - 1200a = 475 + 125\sqrt{5}\end{aligned}$$

So we have

$$\widetilde{Y}_t - a\widetilde{Y}_{t-1} = K^{t-1}\overline{Y}_1 = \lambda \cdot K^t$$

where $\lambda = \overline{Y}_1/K = 875 + 75\sqrt{5}$

$\widetilde{Y}_t \Leftarrow \hat{Y}_t^* \Leftarrow \overline{Y}_t$

Step 4:

$$\widetilde{Y}_t = a\widetilde{Y}_{t-1} + \lambda \cdot K^t$$

To proceed, we need to guess a solution of $\widetilde{Y}_t$. Assume the solution is in the form of $\widetilde{Y}_t^* = C_1 K^t$, then

$$C_1 K^t = a \cdot C_1 K^{t-1} + \lambda \cdot K^t$$

Dividing both sides by K^{t-1}, we obtain

$$KC_1 = aC_1 + \lambda K \quad \Rightarrow C_1 = \frac{\lambda K}{K - a} = 625 + 475\sqrt{5}$$

Write $\hat{Y}_t = \widetilde{Y}_t - \widetilde{Y}_t^* = \widetilde{Y}_t - C_1 K^t$ then

$$\hat{Y}_t = a\hat{Y}_{t-1}, \Rightarrow \widetilde{Y}_t - \widetilde{Y}_t^* = \hat{Y}_t = a^t \hat{Y}_0 = a^t(\widetilde{Y}_0 - C_1)$$

$Y_t \Leftarrow \widetilde{Y}_t \Leftarrow \bar{Y}_t$

Step 5:

$$\widetilde{Y}_t - C_1 K^t = a^t(\widetilde{Y}_0 - C_1) =: a^t \cdot C_2$$

$$\widetilde{Y}_t = C_1 K^t + C_2 \cdot a^t$$

$$Y_t = \widetilde{Y}_t + 500 = C_1 K^t + C_2 \cdot a^t + 500$$

where from above

$$C_2 = Y_1 - 500 - C_1 = 1075 - C_1 = 450 - 475\sqrt{5}$$

Plugging-in all other values C_1, a and K obtained above, we derive the general formula of Y_t.

General Solution

- Consider the second-order linear difference equation

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + b$$

  In our example: $\phi_1 = 1$, $\phi_2 = -0.2$ and $b = 100$.

- Suppose $\phi_1 + \phi_2 \neq 1$ and the characteristic equation is

$$x^2 = \phi_1 x + \phi_2, \text{ or equivalently, } \quad x^2 - \phi_1 x - \phi_2 = 0$$

  which has two different roots a and K.

- Let

$$Y_t = C_1 \cdot a^t + C_2 \cdot K^t + \frac{b}{1 - \phi_1 - \phi_2}$$

  where C_1 and C_2 are arbitrary constants.

Initial Conditions

If initial value Y_0 and Y_1 are given, we can solve out:

$$Y_0 = C_1 + C_2 + \frac{b}{1 - \phi_1 - \phi_2}, \quad Y_1 = C_1 a + C_2 K + \frac{b}{1 - \phi_1 - \phi_2}$$

to derive C_1 and C_2, where a and K are previously derived.

Stability

When $|a| < 1$ and $|K| < 1$, regardless what the initial values Y_0 and Y_1 are, we have

$$\begin{aligned} Y_t =& C_1 \cdot a^t + C_2 \cdot K^t + \frac{b}{1 - \phi_1 - \phi_2} \\ \to& C_1 \cdot 0 + C_2 \cdot 0 + \frac{b}{1 - \phi_1 - \phi_2} \\ =& \frac{b}{1 - \phi_1 - \phi_2}, \end{aligned}$$

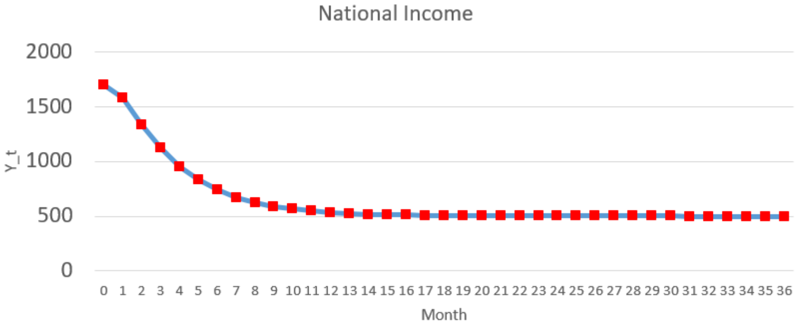
which is the **equilibrium state**.
In our example, we obtain the equilibrium state as

$$b/(1 - \phi_1 - \phi_2) = 100/(1 - 1 - (-0.2)) = 500$$

Time Path: National Income

$$Y_t = Y_{t-1} - 0.2Y_{t-2} + 100, \quad t = 2, 3, \ldots$$

with $a \approx 0.2764$ and $K \approx 0.7236$



A reminder to complete
SETU (Student Evaluation of Teaching and Units)