ETF2700/ETF5970 Mathematics for Business

Lecture 10

Monash Business School, Monash University,
Australia

Outline

Last week (introduction to investment):

- Basic theory of interest
- Cash flow stream: Sequences and series
- Present value techniques and applications

This week:

- Depreciation, inflation, and real growth
- A glimpse of difference equation

Student Evaluation of Teaching and Units, SETU

- All students are encouraged to complete SETU Survey on Moodle
- Your feedback is highly appreciated

Depreciation

- On July 1, 2019, your company purchased an equipment with a cost of $10,500.

- Will you still “report” $10,500 one year later?
- No. The value is **depreciated** due to the use of the equipment, or new technology, etc.
- The decline in the value of an asset is called depreciation.

Mathematics for depreciation

- $A_0 = 10500$: Original *book value* of the equipment
- A_t: The *book value* of the equipment after t years of depreciation
- i: Depreciation rate per year

The *book value* after one year is

$$A_1 = A_0(1 - i) = A_0 - \underbrace{i \cdot A_0}_{\text{depreciated value}}$$

How can we obtain the depreciation rate i? We need further information.

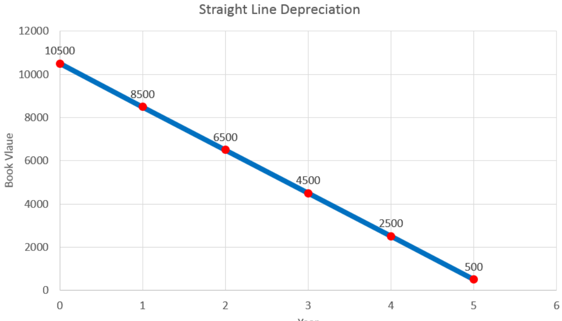
- The equipment will have a useful life of 5 years
- After 5 years, your company expects to sell it for $500
- Mathematically, such information means $A_5 = 500$
- Can we determine the depreciate rate i using such information? Not yet. We need to choose a method.

Straight line depreciation

Assumption: The value decreases by same **amount** each year

- $A_1 = A_0 - i \cdot A_0 = A_0(1-i)$
- $A_2 = A_1 - i \cdot A_0 = A_0(1-2i)$
- $A_3 = A_2 - i \cdot A_0 = A_0(1-3i)$
- $A_t = A_0(1-it)$ after t years of depreciation
- Solve $A_5 = A_0(1-5i)$, that is,

$$500 = 10500(1-5i) \quad \Rightarrow \quad i \approx 0.1905 \Rightarrow A_1 = A_0(1-i) = 8500$$

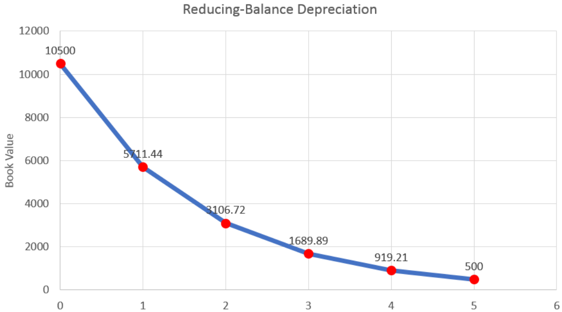


Reducing-balance depreciation

Assumption: The value decreases by same **rate** each year.

- $A_1 = A_0 \cdot (1 - i) = A_0(1 - i)$
- $A_2 = A_1 \cdot (1 - i) = A_0(1 - i)^2$
- $A_3 = A_2 \cdot (1 - i) = A_0(1 - i)^3$
- $A_t = A_0(1 - i)^t$ after t years of depreciation Solve $A_5 = A_0(1 - i)^5$, that is,

$$500 = 10500(1-i)^5 \Rightarrow i \approx 0.4561 \Rightarrow A_1 = A_0(1-i) \approx 5711.44$$



Straight Line vs Reducing Balance

In our example,

- Straight Line: $A_1 = 8500$
- Reducing Balance: $A_1 \approx 5711.44 < 8500$

Straight Line

- an equal amount each period
- most commonly used because of its simplicity

Reducing balance

- more in the early years than in the later years
- depending on the type of asset, you may find this is more appropriate

Example

A machine cost for \$30,000 and is depreciated at 15% p.a. After 5 years, what is its value and the total amount of depreciation?

We have $A_0 = 30000\$$ and $i = 15\%$.

Straight-Line Depreciation

$A_5 = A_0(1 - 5i) = 30000 \cdot (1 - 5 \cdot 0.15) = 7500\$$
$A_5 - A_0 = 30000 - 7500 = 22500\$$

Reducing-balance Depreciation

$A_5 = A_0(1 - i)^5 = 30000 \cdot (1 - 0.15)^5 \approx 13311.16\$$
$A_5 - A_0 \approx 30000 - 13311.16 = 16688.84\$$

Inflation: Depreciation of currency

Suppose now

- you have $P_0 = 100$ dollars cash
- each unit of good sells for 1\$: you can buy 100 units

You do not deposit the cash and one year later

- you still have $P_1 = 100$ dollars cash
- the price increases by $r_i = 25\%$, so each unit sells for 1.25\$: you can buy $\frac{100}{1+r_i} = 80$ units

In terms of purchasing power:

$$100\$ \text{ later} = \frac{1}{1+r_i} \cdot 100 \text{ goods} = 0.8 \cdot 100\$ \text{ now}$$

The dollars depreciated by $i = 1 - 0.8 = 0.2 = 20\%$ p.a.

Real Growth

Suppose you now have P_0 dollars cash and deposit it in the bank

- Receives interest: $P_0(1 + r)$
- All prices increase by inflation rate r_i
- The real value is

$$\frac{P_0(1 + r)}{1 + r_i} = P_0 \cdot \left(\frac{1 + r}{1 + r_i}\right)$$

- The **real growth** r_{real}:

$$P_0 \cdot \left(\frac{1 + r}{1 + r_i}\right) = P_0(1 + r_{\text{real}}) \Rightarrow r_{\text{real}} = \frac{1 + r}{1 + r_i} - 1$$

A Glimpse of Difference Equation

Review: Sequence

In the 9th week lecture, a sequence is an ordered list of numbers

$$T_1, T_2, T_3, T_4, \ldots$$

- Investment Project: T_n is the cash flow at time n

For convenience, in this lecture, we start from Y_0, the list becomes

$$Y_0, Y_1, Y_2, Y_3, Y_4, \ldots, Y_t, \ldots$$

where Y_0 is a (given) "starting value".

Arithmetic sequence

- Recurrence relation: $Y_{t+1} = Y_t + d, \ t = 0, 1, 2, \ldots$
- General Formula: $Y_t = Y_0 + t \cdot d, \ t = 0, 1, 2, \ldots$

Geometric sequence

- Recurrence relation: $Y_{t+1} = K \cdot Y_t, \ t = 0, 1, 2, \ldots$
- General Formula: $Y_t = K^t Y_0, \ t = 0, 1, 2, \ldots$

- Each of the recurrence relations above is a so-called **difference equation**
- The general formula of Y_t is the **solution** to the difference equation: Y_0 is often given.

A simple example

Suppose you have an initial savings $Y_0 = \$1000$ in cash. In every month $t + 1$, $t = 0, 1, \ldots$,

- you spend 80% of your savings in the last month t; and
- you receive an income $\$5000$ in cash

For simplicity, we assume interest rate is $r = 0$.

$$Y_t = \text{your savings in \$ at the end of month } t.$$

We have a recurrence relation

$$Y_{t+1} = (1 - 0.8) \cdot Y_t + 5000, \quad t = 0, 1, \ldots$$

and an initial condition $Y_0 = 1000$.

Your savings (in \$) at the end of month t:

$$Y_{t+1} = 0.2 \cdot Y_t + 5000, \quad t = 0, 1, \ldots$$

with $Y_0 = 1000$.

- It is neither an arithmetic nor a geometric sequence
- $Y_1 = 0.2 \cdot 1000 + 5000 = 5200$
- $Y_2 = 0.2 \cdot 5200 + 5000 = 6040$
- $Y_3 = 0.2 \cdot 6040 + 5000 = 6208$
- $Y_4 = 0.2 \cdot 6208 + 5000 = 6241.6$

$$\vdots$$

First-order linear difference equation

A first-order linear difference equation

$$Y_{t+1} = aY_t + b$$

- a, b and Y_0 are given
- First-order: Y_{t+1} is fully determined by the 1-period lagged value Y_t
- Linear: $Y_{t+1} = f(Y_t)$, where $f(x) = ax + b$ is a linear function
- Our example: $a = 0.2$ and $b = 5000$.
- If $a = 1$, the sequence would be an arithmetic sequence $Y_t = Y_0 + t \cdot b$

Solve our example

From sequence equation to difference equation

$$Y_{t+1} = 0.2 \cdot Y_t + 5000, \quad t = 0, 1, \ldots$$

with $Y_0 = 1000$. Can we solve Y_t, for all $t = 1, 2, \ldots$?

1) Rewrite the difference equation

$$\underbrace{Y_{t+1} - 6250}_{\widetilde{Y}_{t+1}} = 0.2 \cdot \underbrace{(Y_t - 6250)}_{\widetilde{Y}_t}$$

2) Let $\widetilde{Y}_t = Y_t - 6250$, which is a geometric sequence:

$$\widetilde{Y}_t = 0.2^t \widetilde{Y}_0 = 0.2^t \cdot (1000 - 6250) = -5250 \cdot 0.2^t$$

Example continued

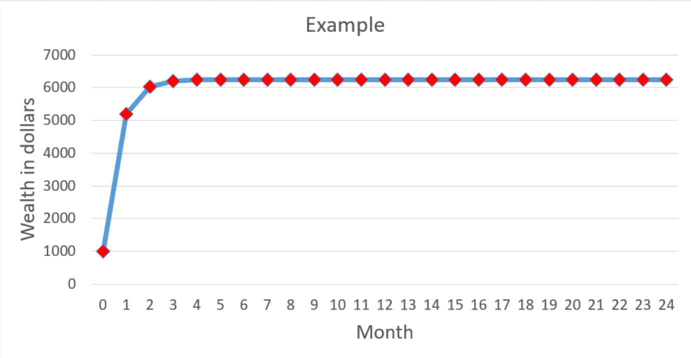
3) Write $Y_t = \widetilde{Y}_t + 6250 = -5250 \cdot 0.2^t + 6250.$

Now we can compute all Y_t's:

- $Y_1 = -5250 \times 0.2 + 6250 = -1050 + 6250 = 5200$
- $Y_2 = -5250 \times 0.04 + 6250 = -210 + 6250 = 6040$
- $Y_3 = -5250 \times 0.008 + 6250 = -42 + 6250 = 6208$
- $\dots$
- $Y_{12} = -5250 \times 0.000000004096 + 6250 \approx 6249.99998$
- $\dots$

$Y_{t+1} = 0.2Y_t + 5000$ with $Y_0 = 1000$

t	Y_t
0	1000
1	5200
2	6040
3	6208
4	6241.6
5	6248.32
6	6249.664
7	6249.933
8	6249.987
9	6249.997
10	6249.999
11	6250
12	6250
13	6250
14	6250
15	6250
16	6250
17	6250
18	6250
19	6250
20	6250
21	6250
22	6250
23	6250
24	6250



Equilibrium State: Example

- In Excel sheet: We find that $Y_t = 6250$ for $t \geq 11$
- This is **not true** mathematically: Y_t is simply too close to 6250, and Excel has rounded off the error.
- After 11 months, your wealth will remain almost constant at the level 6250.
- Precisely, as $t \to \infty$, $0.2^t \to 0$ and thus

$$
\begin{aligned}
Y_t &= -5250 \cdot 0.2^t + 6250 \\
&\to -5250 \cdot 0 + 6250 = 6250
\end{aligned}
$$

We call this level \$6250 an **equilibrium state**, or a **stationary state**, of the difference equation in this example.

Solving $Y_{t+1} = aY_t + b$ with $a \neq 1$

Rewrite the difference equation

$$Y_{t+1} - C = a \cdot (Y_t - C)$$

with some constant C. What is the value of C?

The above difference equation is equivalent to

$$Y_{t+1} = a \cdot Y_t + (1 - a)C$$

Noting that $1 - a \neq 0$, so

$$(1 - a)C = b \quad \Leftrightarrow \quad C = \frac{b}{1 - a}$$

Solving $Y_{t+1} = aY_t + b$ with $a \neq 1$

1) Rewrite the difference equation

$$\underbrace{Y_{t+1} - \frac{b}{1-a}}_{\widetilde{Y}_{t+1}} = a \cdot (\underbrace{Y_t - \frac{b}{1-a}}_{\widetilde{Y}_t})$$

2) $\widetilde{Y}_t = Y_t - \frac{b}{1-a}$ is a geometric sequence

$$\widetilde{Y}_t = a^t \widetilde{Y}_0 = a^t \left(Y_0 - \frac{b}{1-a} \right)$$

3) Write $Y_t = \widetilde{Y}_t + \frac{b}{1-a} = a^t \left(Y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$

Solution: $Y_{t+1} = aY_t + b$

When $a \neq 1$,

$$Y_t = \widetilde{Y}_t + \frac{b}{1-a} = a^t \left(Y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$$

In our example $a = 0.2$, $b = 5000$, $Y_0 = 1000$, so

$$\begin{aligned} Y_t =& 0.2^t \left(1000 - \frac{5000}{1-0.2} \right) + \frac{5000}{1-0.2} \\ =& -5250 \cdot 0.2^t + 6250 \end{aligned}$$

Equilibrium State: $Y_{t+1} = aY_t + b$

A point y^* is an equilibrium/stationary state if

$$Y_{t_0} = y^* \quad \text{and} \quad Y_t = y^* \text{ for all } t \geq t_0$$

Recall $Y_{t+1} = f(Y_t)$ with $f(y) = ay + b$. An equilibrium/stationary state y^* is a solution of the equation

$$y = f(y) \quad \Leftrightarrow \quad y = ay + b$$

- When $a \neq 1$: $y^* = \frac{b}{1-a}$
- When $a = 1$ but $b \neq 0$: no equilibrium state
- When $a = 1$ but $b = 0$: any $y^* \in (-\infty, \infty)$

The difference equation

$$Y_{t+1} = aY_t + b, \quad a \neq 1,$$

has only an equilibrium state $y^* = \frac{b}{1-a}$.

When $Y_0 = \frac{b}{1-a}$

- $Y_1 = aY_0 + b = a \cdot \frac{b}{1-a} + b = b \cdot \frac{a+(1-a)}{1-a} = \frac{b}{1-a}$
- $Y_2 = aY_1 + b = a \cdot \frac{b}{1-a} + b = b \cdot \frac{a+(1-a)}{1-a} = \frac{b}{1-a}$

When $Y_0 \neq \frac{b}{1-a}$

- As $t \to \infty$, can we have $Y_t \to \frac{b}{1-a}$?
- Does Y_t converge to the equilibrium state $\frac{b}{1-a}$ in long run?
- Recall that

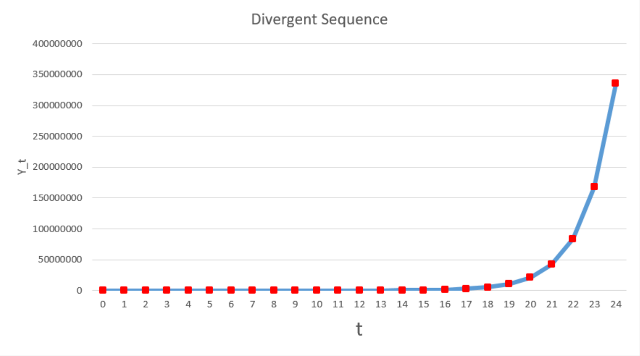
$$Y_t = a^t \left(Y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$$

Yes if $a^t \to 0$ that means $|a| < 1$. The difference equation is **globally asymptotically stable**, or sometimes **stable**.

An Example: A “divergent” sequence

$$Y_{t+1} = 2Y_t + 10 \text{ with } Y_0 = 10$$

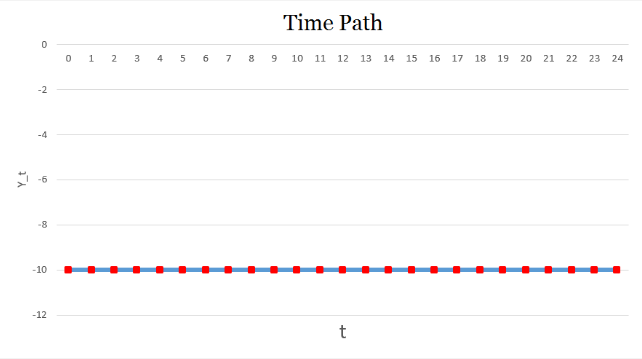
t	Y_t
0	10
1	30
2	70
3	150
4	310
5	630
6	1270
7	2550
8	5110
9	10230
10	20470
11	40950
12	81910
13	163830
14	327670
15	655350
16	1310710
17	2621430
18	5242870
19	10485750
20	20971510
21	41943030
22	83886070
23	167772150
24	335544310



An Example: A “constant” sequence

$$Y_{t+1} = 2Y_t + 10 \text{ with } Y_0 = -10$$

t	Y_t
0	-10
1	-10
2	-10
3	-10
4	-10
5	-10
6	-10
7	-10
8	-10
9	-10
10	-10
11	-10
12	-10
13	-10
14	-10
15	-10
16	-10
17	-10
18	-10
19	-10
20	-10
21	-10
22	-10
23	-10
24	-10



An Example: Convergence with oscillation

$$Y_{t+1} = -\frac{1}{2}Y_t + 10 \text{ with } Y_0 = 10$$