More complicated:

$\rho_0 =$ constant

$\vec{B}_0 = B_0 \hat{z}$ $\Rightarrow$ $\nabla \times \vec{B}_0 = 0$, $\nabla p = 0$, $\vec{g} = 0$

(removes isotropy)

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - \gamma p_0 \nabla \nabla \cdot \vec{u} - \frac{1}{\mu_0}\left(\nabla \times \nabla \times (\vec{u} \times \vec{B}_0)\right) \times \vec{B}_0 = 0$$

Substitute Fourier-derivatives:

$$-\rho_0 \omega^2 \vec{u} - \gamma p_0 (-i\vec{k})(-i\vec{k}\cdot\vec{u}) - \frac{1}{\mu_0}\left((-i\vec{k}) \times (-i\vec{k}) \times (\vec{u} \times B_0 \vec{z})\right) \times B_0 \vec{z} = 0$$

$$-\rho_0 \omega^2 \vec{u} + \gamma p_0 \vec{k}(\vec{k}\cdot\vec{u}) + \frac{B_0^2}{\mu_0}\left(\vec{k} \times (\vec{k} \times (\vec{u} \times \vec{z}))\right) \times \vec{z} = 0. \quad / \rho_0$$

$$-\omega^2 \vec{u} + \overbrace{\frac{\gamma p_0}{\rho_0}}^{c_s^2} \vec{k}(\vec{k}\cdot\vec{u}) + \overbrace{\frac{B_0^2}{\rho_0 \mu_0}}^{V_A^2 \text{ - Alfven speed}} \left(\vec{k} \times (\vec{k} \times (\vec{u} \times \vec{z}))\right) \times \vec{z} = 0$$

If we multiply it scalarly by $\vec{z}\cdot$, the last term is 0 since last $\times \vec{z}$ makes it perpendicular to $\vec{z}$

$$\cdot\vec{z}: -\omega^2 u_z + \underbrace{\frac{\gamma p}{\rho}}_{= c_s^2} m (\vec{k}\cdot\vec{u}) = 0.$$

m is z-component of $\vec{k}$

$\vec{k} = (k, \ell, m)$

$\vec{k}\cdot\vec{k} = K^2$

$$\boxed{-\omega^2 u_z + c_s^2 m (\vec{k}\cdot\vec{u}) = 0.}$$

We also scalarly multiply it by $\vec{k}\cdot$

$$-\omega^2 (\vec{k}\cdot\vec{u}) + c_s^2 K^2 (\vec{k}\cdot\vec{u}) + V_A^2 \vec{k}\cdot\left[\left(\vec{k} \times (\vec{k} \times (\vec{u} \times \vec{z}))\right) \times \vec{z}\right] = 0.$$

After some mess (use $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$):

$$\boxed{-\omega^2 (\vec{k}\cdot\vec{u}) + c_s^2 K^2 (\vec{k}\cdot\vec{u}) + V_A^2 K^2 (\vec{k}\cdot\vec{u}) - m K^2 V_A^2 u_z = 0.}$$

$$\begin{cases} -\omega^2 u_z + c_s^2 m(\vec{K}\cdot\vec{u}) = 0 \\ -mK^2V_A^2 u_z - (\vec{K}\cdot\vec{u})\left(\omega^2 - (c_s^2+V_A^2)K^2\right) = 0. \end{cases}$$

u_z and $(\vec{K}\cdot\vec{u})$ are unknown.
Linear system, homogeneous.
Has non-trivial solutions when $\det = 0$.

$$\det = \omega^2\left(\omega^2 - (c_s^2+V_A^2)K^2\right) - c_s^2 m^2 K^2 V_A^2 = 0$$

$$\boxed{\omega^4 - \omega^2 K^2 (c_s^2 + V_A^2) - c_s^2 m^2 K^2 V_A^2 = 0.}$$

Dispersion relation for
all magneto-acoustic waves.

Solution:

$$\omega^2 = \frac{K^2}{2}(c_s^2+V_a^2) \pm \frac{K^2}{2}\sqrt{(c_s^2+V_A^2)^2 - \frac{c_s^2 m^2 V_A^2}{K^2}}$$

now, m is in direction of z
the angle between $\vec{k}$ and m is θ
$m = K\cos\theta$

$$\text{so} \quad \omega^2 = \frac{K^2}{2}(c_s^2+v_a^2) \pm \frac{K^2}{2}\sqrt{(c_s^2+v_a^2)^2 - 4c_s^2V_A^2\cos^2\theta}$$

$\pm$ — two solutions

$+$ is called "fast magnetoacoustic wave"

$-$ is called "slow magnetoacoustic wave"

Depends on angle

$\theta = 0 \quad \cos\theta = 1$

$$V_{ph}^2 = \frac{\omega^2}{K^2} = \frac{C_s^2 + V_a^2}{2} \pm \frac{1}{2}\sqrt{(C_s^2+V_a^2)^2 - 4C_s^2V_a^2} =$$

$$= \frac{C_s^2+V_a^2}{2} \pm \frac{\sqrt{(C_s^2-V_a^2)^2}}{2} = C_s^2 \text{ or } V_A^2$$

Depending on $C_s^2 >$ or $< V_a^2$

$$\text{Slow} \quad C_{ph}^2 = \min(C_s^2, V_a^2)$$

$$\text{Fast} \quad C_{ph}^2 = \max(C_s^2, V_a^2)$$

$\theta = \frac{\pi}{2} \quad \cos\theta = 0$

$$\text{Slow} \quad C_{ph}^2 = 0$$

$$\text{Fast} \quad C_{ph}^2 = C_s^2 + V_a^2$$

Slow wave does not propagate across magnetic field

$$\underbrace{-\omega^2\vec{u} + c_s^2 \vec{k}(\vec{k}\cdot\vec{u}) + V_a^2\left(\vec{k}\times\left(\vec{k}\times\left(\vec{u}\times\vec{z}\right)\right)\right)\times\vec{z} = 0}$$

$$\vec{k}\cdot\left(\quad \times\vec{z}\right) =$$

$$\omega^2\left(\vec{k}\cdot(\vec{u}\times\vec{z})\right) - \cancel{c_s^2(\vec{k}\cdot\vec{u})\vec{k}\cdot(\vec{k}\times\vec{z})} - V_a^2 m^2\left(\vec{k}\cdot(\vec{u}\times\vec{z})\right) +$$

$$+ \cancel{V_a^2 m u_z \vec{k}\cdot(\vec{k}\times\vec{z})} + \cancel{V_a^2(\vec{k}\cdot\vec{u}) m\vec{k}(\vec{z}\times\vec{z})} + \cancel{V_a^2(\vec{k}\cdot\vec{u})\vec{k}\cdot(\vec{k}\times\vec{z})}$$

Some more messy derivations lead to:

$$(\omega^2 - m^2 V_a^2)\left(\vec{k}\cdot(\vec{u}\times\vec{z})\right) = 0$$

$\boxed{\omega^2 = m^2 V_a^2}$ Alfvén wave

Does not have c_s^2 inside.
Does not feel pressure or density,
only magnetic field.

$m = k\cos\theta$

$V_{ph}^2 = \dfrac{\omega^2}{k^2} = V_a^2\cos^2\theta$ propagates along magnetic field.

$\vec{k}\cdot(\vec{u}\times\vec{z}) \Rightarrow u$ is perpendicular to both $\vec{k}$ and $\vec{z}$. Transverse waves!