We are looking at small perturbations around some equilibrium state, defined as (subscript 0):

$$\nabla\left(p_0 + \frac{B_0^2}{2\mu_0}\right) - \frac{1}{\mu_0}(\vec{B}_0\cdot\nabla)\vec{B}_0 = \rho_0\vec{g}$$

or

$$\nabla p_0 - \vec{J}_0\times\vec{B}_0 = \rho_0\vec{g} \quad , \quad \vec{J}_0 = \frac{1}{\mu_0}\nabla\times\vec{B}_0$$

No energy losses, adiabatic process assumed.

$$p = C\rho^\gamma \; ; \quad \frac{p}{\rho^\gamma} = C \; ; \quad \frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

Instead of using general MHD energy eq. we use this simplified one.

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{1}{\rho^\gamma}\frac{Dp}{Dt} + p\frac{D}{Dt}\rho^{-\gamma} =$$

$$= \frac{1}{\rho^\gamma}\frac{Dp}{Dt} + p\cdot(-\gamma)\cdot\rho^{-\gamma-1}\frac{D\rho}{Dt} = \frac{1}{\rho^\gamma}\frac{Dp}{Dt} - p\gamma\frac{1}{\rho^{\gamma+1}}\frac{D\rho}{Dt} =$$

$$\left(\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\nabla)\right)$$

$$= \frac{1}{\rho^\gamma}\frac{\partial p}{\partial t} + \frac{1}{\rho^\gamma}(\vec{V}\cdot\nabla)p - \frac{p\gamma}{\rho^{\gamma+1}}\frac{\partial\rho}{\partial t} - \frac{p\gamma}{\rho^{\gamma+1}}(\vec{V}\cdot\nabla)\rho = 0.$$

$$\frac{\partial p}{\partial t} + (\vec{V}\cdot\nabla)p - \frac{p\gamma}{\rho}\left(\frac{\partial\rho}{\partial t} + (\vec{V}\cdot\nabla)\rho\right) = 0$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho V) = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + (V\cdot\nabla)\rho + \rho\nabla\cdot V = 0.$$

$$\frac{\partial\rho}{\partial t} + (V\cdot\nabla)\rho = -\rho\nabla\cdot V$$

$$\frac{\partial p}{\partial t} + (\vec{V}\cdot\nabla)p + \frac{p\gamma}{\cancel{\rho}}\cancel{\rho}\nabla\cdot V = 0.$$

$$\boxed{\frac{\partial p}{\partial t} + (V\cdot\nabla)p + p\gamma\nabla\cdot V = 0}$$

Now our MHD system is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u}\vec{u}) + \nabla p - \vec{J} \times \vec{B} = \rho \vec{g} \qquad (2)$$

$$\frac{\partial p}{\partial t} + (\vec{v} \cdot \nabla) p + p\gamma \nabla \cdot \vec{v} = 0$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$

(2) can be rewritten ~~us~~ using

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u}\vec{u}) = \underbrace{\vec{u}\frac{\partial \rho}{\partial t}} + \rho \frac{\partial \vec{u}}{\partial t} + (\rho \vec{u} \cdot \nabla)\vec{u} + \underbrace{\vec{u}(\nabla \cdot \rho \vec{u})}_{=0 \text{ (cont. eq.)}}$$

$$(2) \Rightarrow \quad \rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla)\vec{u} + \nabla p - \vec{J} \times \vec{B} = \rho \vec{g}$$

Assume small perturbations to $\rho, p, \vec{v}, \vec{B}$:

$$\vec{B} = \vec{B}_0 + \vec{B}_1(\vec{r}, t)$$
$$\vec{v} = \vec{v}_0 + \vec{v}_1(\vec{r}, t)$$
$$\rho = \rho_0 + \rho_1(\vec{r}, t)$$
$$p = \underbrace{p_0}_{\text{equilibrium}} + \underbrace{p_1(\vec{r}, t)}_{\text{perturbed}}$$

Note: product of two small terms is 0.

0-index variables do not depend on time —
static initial cons.

$\vec{v}_0 = 0$.

Substitute into the MHD system:

CE: $$\cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho_1}{\partial t} + \nabla\cdot\left(\left(\rho_0 + \cancelto{0}{\rho_1}\right)\vec{v}\right) =$$

$$= \boxed{\frac{\partial \rho_1}{\partial t} + \nabla\left(\rho_0 \vec{V}_1\right) = 0}$$

ME:

$$\left(\rho_0 + \underbrace{\rho_1\right)\frac{\partial \vec{u}_1}{\partial t}}_{\text{small} = 0} + \left(\rho_0 + \rho_1\right)\underbrace{\left(\vec{u}_1 \cdot \nabla\right)\vec{u}_1}_{\text{small} = 0} + \nabla\left(p_0 + p_1\right) - \frac{1}{\mu_0}\left(\nabla \times \left(\vec{B}_0 + \vec{B}_1\right)\right) \times \left(\vec{B}_0 + \vec{B}_1\right) =$$

$$= \left(\rho_0 + \rho_1\right)\vec{g}$$

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} + \underbrace{\nabla p_0} + \nabla p_1 - \underbrace{\frac{1}{\mu_0}\left(\nabla \times \vec{B}_0\right) \times \vec{B}_0} - \underbrace{\frac{1}{\mu_0}\left(\nabla \times \vec{B}_1\right) \times \vec{B}_1}_{\text{small} = 0} - \frac{1}{\mu_0}\left(\nabla \times \vec{B}_0\right) \times \vec{B}_1 -$$

$$- \frac{1}{\mu_0}\left(\nabla \times \vec{B}_1\right) \times \vec{B}_0 = \underbrace{\rho_0 \vec{g}} + \rho_1 \vec{g}$$

These are our initial equilibrium $= 0$.

$$\boxed{\rho_0 \frac{\partial u}{\partial t} + \nabla p_1 - \frac{1}{\mu_0}\left(\nabla \times \vec{B}_0\right) \times \vec{B}_1 - \frac{1}{\mu_0}\left(\nabla \times \vec{B}_1\right) \times \vec{B}_0 = \rho_1 \vec{g}}$$

PE: $$\frac{\partial\left(p_0 + p_1\right)}{\underbrace{\partial t}_{=0}} + \underbrace{\left(v \cdot \nabla\right)\left(p_0 + p_1\right)}_{=0} + \left(p_0 + \underbrace{p_1\right)\gamma \nabla \cdot v}_{=0} = 0.$$

$$\boxed{\frac{\partial p_1}{\partial t} + \left(v \cdot \nabla\right)p_0 + \gamma p_0 \nabla \cdot v = 0}$$

IE: $$\frac{\partial\left(\vec{B}_0 + \vec{B}_1\right)}{\partial t} = \nabla \times \left(\vec{v} \times \left(\vec{B}_0 + \vec{B}_1\right)\right)$$

$$\Rightarrow \boxed{\frac{\partial \vec{B}_1}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B}_0\right)}$$

Take $\frac{\partial}{\partial t}$ of ME:

$$\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} + \nabla \frac{\partial p}{\partial t} - \frac{1}{\mu_0}\left(\nabla \times \frac{\partial \vec{B}_1}{\partial t}\right) \times \vec{B}_0 - \frac{1}{\mu_0}(\nabla \times \vec{B}_0) \times \frac{\partial \vec{B}_1}{\partial t} = \frac{\partial \rho_1}{\partial t}\vec{g}$$

Substitute $\frac{\partial p}{\partial t}, \frac{\partial B_1}{\partial t}, \frac{\partial \rho_1}{\partial t}$:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - \nabla\left((\vec{v} \cdot \nabla) p_0 + \gamma p_0 \nabla \cdot \vec{u}\right) - \frac{1}{\mu_0}\left(\nabla \times \nabla \times (\vec{u} \times \vec{B}_0)\right) \times \vec{B}_0 -$$

$$- \frac{1}{\mu_0}\left(\nabla \times \vec{B}_0\right) \times \nabla \times \left(\vec{u} \times \vec{B}_0\right) = \nabla \cdot \left(\rho_0 \vec{u}\right) \vec{g}$$

1 equation, 1 unknown!

Linearized equation of motion. Basis for MHD waves and instabilities.

Start from simplest:

$\vec{g} = 0$

$\vec{B}_0 = 0$

$\nabla p_0 = 0$, $p_0 = \text{const}$, $\rho_0 = \text{const}$.

Very high β plasma, short lengths
(shorter than the pressure scale)

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - \gamma p_0 \nabla(\nabla \cdot u) = 0 .$$

Take divergence $\nabla\cdot$

$$\rho_0 \frac{\partial^2 (\nabla \cdot u)}{\partial t^2} - \gamma p_0 \nabla^2 (\nabla \cdot u) = 0$$

$$\delta = (\nabla \cdot u)$$

$$\rho_0 \frac{\partial^2 \delta}{\partial t^2} - \gamma p_0 \nabla^2 \delta = 0$$

or $$\frac{\partial^2 \delta}{\partial t^2} - \frac{\gamma p_0}{\rho_0} \nabla^2 \delta = 0 .$$ Wave equation.

Characteristic speed $c_s^2 = \dfrac{\gamma p_0}{\rho_0}$ –

– Sound speed. δ is not zero $\Rightarrow$
compressional waves.

$$\frac{\partial^2 \delta}{\partial t^2} - c_s^2 \nabla^2 \delta = 0. \qquad (*)$$

Fourier analysis.

$$\delta = \delta_0 \exp\left(i(\omega t - \vec{k}\cdot\vec{r})\right)$$

$\vec{k}\cdot\vec{r} = k_x \cdot x + k_y y + k_z z = k\cdot x + l\cdot y + m\cdot z$

ω - frequency

$\vec{k}$ - wave vector

$\vec{r}$ - position vector.

$$\frac{\partial \delta}{\partial t} = i\omega \;; \quad \frac{\partial^2 \delta}{\partial t^2} = -\omega^2$$

$$\frac{\partial \delta}{\partial x} = -ik \;; \quad \frac{\partial^2 \delta}{\partial x^2} = -k^2$$

In general, $\nabla\cdot = -i\vec{k}\cdot \;, \quad \nabla = -i\vec{k} \;; \quad \nabla\times = -i\vec{k}\times$

Apply it to $(*)$;

$$-\omega^2 \delta + c_s^2 K^2 \delta = 0 \;, \quad \text{where } K^2 = k^2 + l^2 + m^2$$

Or $(\omega^2 - c_s^2 K^2)\delta = 0.$

Either $\delta = 0$, which is trivial

Or $\omega^2 = c_s^2 K^2$

$$\boxed{\omega^2 = c_s^2 K^2}$$

Dispersion relation for sound waves.

DR, $\omega = \omega(\vec{k})$, can be used to define phase and group speeds.

Phase speed $\quad c_{ph} = \frac{\omega}{K} = \pm c_s$ $\quad$ is the sound speed

Group speed $\quad \vec{c}_g = \frac{\partial \omega}{\partial \vec{K}} = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right)$

$$\omega^2 = (k^2 + l^2 + m^2)\, c_s^2 \quad \Big| \frac{\partial}{\partial \vec{K}}$$

$$2\omega \frac{\partial \omega}{\partial \vec{K}} = \left(2k c_s^2, 2l c_s^2, 2m c_s^2 \right)$$

$$\vec{c}_g = \left(c_s^2 \frac{k}{\omega}, c_s^2 \frac{l}{\omega}, c_s^2 \frac{m}{\omega} \right) = c_s^2 \frac{K}{\omega} \hat{\vec{K}} = c_s \cdot \hat{\vec{K}}$$

Direction of energy transport. $\quad$ $\hat{\vec{K}}$ - unit vector in direction of $\vec{K}$