MHD applications:

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla\cdot\left(\rho\vec{u}\vec{u} + \left(p + \frac{B^2}{2\mu_0}\right)\vec{I} - \frac{1}{\mu_0}\vec{B}\vec{B}\right) = \rho g$$

Magneto-hydrostatics:

Long-lived structures on the Sun

$\vec{u} = 0$

$\frac{\partial \rho\vec{u}}{\partial t} = 0$; $\nabla\cdot\rho\vec{u}\vec{u} = 0$

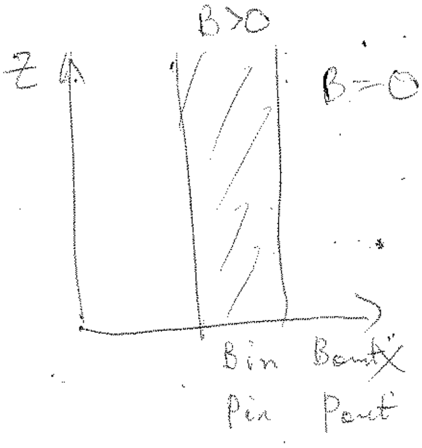
$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) - \frac{1}{\mu_0}\nabla\cdot\vec{B}\vec{B} = \rho\vec{g} \qquad \left(\nabla p - \vec{J}\times\vec{B} = \rho\vec{g}\right)$$

$$\nabla\cdot BB = (\vec{B}\cdot\nabla)\vec{B} + \cancel{(\nabla\cdot\vec{B})\,B} \to 0 \qquad \nabla\cdot\vec{B} = 0$$

magnetic pressure

$$\underbrace{\nabla\left(p + \frac{B^2}{2\mu_0}\right)}_{\text{total pressure}} - \underbrace{\frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B}}_{\text{magnetic tension}} = \rho\vec{g}$$

1) Vertical uniform slab of magnetic field:



$(\vec{B}\cdot\nabla)\vec{B} = 0$

Equilibrium:

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \rho g$$

$B_z = B_z(x)$; $\frac{\partial}{\partial x}\left(p + \frac{B^2}{2\mu_0}\right) = 0$.

$$\frac{B_{in}^2}{2\mu_0} + p_{in} = \frac{B_{out}^2}{2\mu_0} + p_{out}$$

$B_{out} = 0$ $\quad \frac{B_{in}^2}{2\mu_0} + p_{in} = p_{out}$

$p_{in} < p_{out}$

Plasma β:

Compare the Lorentz force with the pressure gradient force:

$$\frac{p}{L} = \frac{B^2}{2\mu_0 L} \qquad \frac{2\mu_0 p}{B^2}$$

If $\frac{2\mu_0 p}{B^2} \gg 1$ — neglect Lorentz force

$\frac{2\mu_0 p}{B^2} \ll 1$ — neglect pres. grad force.

$$\boxed{\beta = \frac{2\mu_0 p}{B^2}}$$

very important quantity in MHD: shows MHD/HD behaviour.

Low-β plasma, MHS equilibrium is just $\vec{J} \times \vec{B} = 0$.

force-free magn. field.

$$\nabla p - \vec{J} \times \vec{B} = \rho \vec{g}$$

What if we selected such a magnetic field that $J \times B = 0$: then plasma does not feel it, and MHS equation reduces to $\nabla p = \rho \vec{g}$ (HS equation).

$$\vec{J} \times \vec{B} = 0. \qquad \vec{J} = \nabla \times \vec{B}$$

$\vec{J} = 0$ – simplest.

$$\nabla \times \vec{B} = 0$$

General solution: $\vec{B} = \nabla \phi$.

ϕ is magnetic scalar potential.

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \cdot \nabla \phi = 0 \qquad \boxed{\nabla^2 \phi = 0}$$

Laplace equation.

Allows separable solutions:

2D, $x-y$ plane

Boundary conditions:

$$\phi(x,0) = F(x)\,;\ \phi(0,y) = \phi(\ell,y) = 0\,;\ \phi \to 0,\ z \to \infty$$

$$\phi = X(x)\,Y(y)\,; \quad \nabla^2 \phi = X''Y + Y''X = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$$

$$Y'' = k^2 Y$$

$$Y = a e^{-ky} + \underline{b e^{ky}}$$

$$X'' = -k^2 X$$

$$X = c \sin kx + \underline{d \cos kx}$$

Applying BC's:

$$\sin kb = 0 \; ; \quad k = \frac{n\pi}{l}$$

$$b = d = 0.$$

Full solution is sum of all possible solutions.

$$\phi = XY = \sum_k \underbrace{A_k \sin kx}_{F} \, e^{-ky}$$

$$A_k = ac$$

$$k = \frac{n\pi}{l}$$

Keep it simple:

F is one Fourier component

$$F(x) = \sin \frac{\pi x}{l}$$

$$\phi(x, y) = \sin \frac{\pi x}{l} e^{-\pi y / l}$$

$$B = \nabla \phi \; ; \quad B_x = B_0 \cos \frac{\pi x}{l} e^{-\pi y / l}$$

$$B_y = -B_0 \sin \frac{\pi x}{l} e^{-\pi y / l}$$

Can be more complex, of course.

$$\vec{J} \times \vec{B} = 0$$

$\vec{J} \neq 0 \Rightarrow$ $\vec{J}$ is parallel to $\vec{B}$

$$\boxed{\mu \vec{J} = \alpha \vec{B}} \qquad \nabla \times \vec{B} = \alpha \vec{B}$$

Schlüter & Temesvary 1958:

$$B_x = -\frac{\partial f}{\partial z} G(f)$$

$$B_z = \frac{\partial f}{\partial x} G(f)$$

$$f = x \cdot B_{0z}(z)$$