Multistep methods, Adams-Bashforth

Note: they have to be started: not self starting as RK.

Advantage - they use previously calculated time steps. So, for one timestep only one calculation is needed, compared to $N(4-5-6\ldots)$ in R-K methods.

Disadvantage - they are less stable.

We derive explicit (Adams-Bashforth), but there are also implicit ones (Adams-Moulton).

IVP: $\frac{dy}{dt} = f(y(t))$

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(y(t))\,dt$$

The idea behind Adams metods is to approximate RHS integral using polynomial interpolation of f at the points $t_{n+1-s}, t_{n+2-s} \ldots t_n$ (explicit).

The interpolation polynomial is on s points, and of $s-1$ order.

We will use Lagrange polynomials - natural choice for this kind of problems, when pairs (x_0, y_0), $(x_1, y_1) \ldots (x_k, y_k)$ are given.

Lagrange* polynomials:

*(first discovered by Edward Waring)

Set of points $(x_0, y_0), (x_1, y_1) \dots (x_K, y_K)$ (grid defined)

(1*) $L(x) = \sum_{j=0}^{K} y_j \, l_j(x)$ — L. interpolation polynomial

where

l_j — L. basis polynomials.

$$l_j(x) = \prod_{\substack{0 \le m \le K \\ m \ne j}} \frac{x - x_m}{x_j - x_m}$$

k factors, so l are k-order polynomials.

Property:

$$l_{j \ne i}(x_i) = \prod_{\substack{0 \le m \le K \\ m \ne j}} \frac{x_i - x_m}{x_j - x_m} = 0$$

($m = i$ gives $x_i - x_i = 0$)

$$l_i(x_i) = \prod_{\substack{0 \le m \le K \\ m \ne i}} \frac{x_i - x_m}{x_i - x_m} = 1;$$

$$\boxed{y_i \, l_i(x_i) = y_i}$$

Therefore, the sum (1*) exactly interpolates the function defined by the set of the points.

Use it now.

2-step A-B method: IVP $\frac{dy}{dt} = f(t, y(t))$

$$f(t_n, y(t_n)) = f_n$$

Given at $(t_{n-1}, f_{n-1}), (t_n, f_n)$

find (t_{n+1}, f_{n+1}), y_{n+1}.

Assume $t_{n+1} - t_n = t_n - t_{n-1} = \Delta t$

1st order L.- interpolation polynomial

$$P(t) = f_n \frac{t - t_{n-1}}{t_n - t_{n-1}} + f_{n-1} \frac{t - t_n}{t_{n-1} - t_n}.$$

Extrapolate onto $n+1$-th step:

$$\int_{t_n}^{t_{n+1}} f(y(t))\, dt = \int_{t_n}^{t_{n+1}} P(t)\, dt.$$

Substitute:

$$\int_{t_n}^{t_{n+1}} \left[f_n \frac{t - t_{n-1}}{t_n - t_{n-1}} + f_{n-1} \frac{t - t_n}{t_{n-1} - t_n} \right] dt =$$

$$= \int_{t_n}^{t_{n+1}} \left[f_n \frac{t - t_{n-1}}{\underbrace{t_n - t_{n-1}}_{\Delta t}} - f_{n-1} \frac{t - t_n}{\underbrace{t_n - t_{n-1}}_{\Delta t}} \right] dt =$$

$$\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left[f_n t - f_{n-1} t - f_n t_{n-1} + f_{n-1} t_n \right] dt =$$

$$= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} t\left(f_n - f_{n-1}\right)dt + \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(f_{n-1}t_n - f_n t_{n-1}\right)dt =$$

$$= \frac{1}{\Delta t}\,\frac{t_{n+1}^2 - t_n^2}{2}\left(f_n - f_{n-1}\right) + \frac{1}{\Delta t}\left(f_{n-1}t_n - f_n t_{n-1}\right)\underbrace{\left(t_{n+1} - t_n\right)}_{\Delta t} =$$

$$= \frac{\left(t_{n+1}+t_n\right)\left(f_n - f_{n-1}\right)}{2} + \left(f_{n-1}t_n - f_n t_{n-1}\right) =$$

$$= \frac{1}{2}t_{n+1}f_n + \frac{1}{2}t_n f_n - \frac{1}{2}t_{n+1}f_{n-1} - \frac{1}{2}t_n f_{n-1} + f_{n-1}t_n - f_n t_{n-1} =$$

$$= f_n\left(\frac{1}{2}t_{n+1} + \frac{1}{2}t_n - t_{n-1}\right) - f_{n-1}\left(\frac{1}{2}t_{n+1} + \frac{1}{2}t_n - t_n\right) =$$

$$= \frac{1}{2}f_n\left(\underbrace{t_{n+1} + \underbrace{t_n - t_{n-1}}_{\Delta t} - t_{n-1}}_{2\Delta t}\right) - \frac{1}{2}f_{n-1}\left(t_{n+1} - t_n\right) =$$

$$= \frac{3}{2}f_n\Delta t - \frac{1}{2}f_{n-1}\Delta t = \frac{\Delta t}{2}\left(3f_n - f_{n-1}\right)$$

So, $y_{n+1} = y_n + \frac{\Delta t}{2}\left(3f_n - f_{n-1}\right)$ $\boxed{AB2}$

Higher orders are derived in the same fashion, but the integrals are longer.

AB3

$$y_{n+1} = y_n + \frac{\Delta t}{12}\left(23 f_n - 16 f_{n-1} + 5 f_{n-2}\right)$$

AB4

$$y_{n+1} = y_n + \left(\frac{55}{24} f_n - \frac{59}{24} f_{n-1} + \frac{37}{24} f_{n-2} - \frac{3}{8} f_{n-3}\right)$$

AB5

$$y_{n+1} = y_n + \left(\frac{1901}{720} f_n - \frac{1387}{360} f_{n-1} + \frac{109}{30} f_{n-2} - \frac{637}{360} f_{n-3} + \frac{251}{720} f_{n-4}\right)$$

- AB is linear, as a linear combination of the previous steps is used.
- Bashforth has nothing to do with numerics, he was HD-mathematician. Adams developed both explicit & implicit methods.
- Non-linear multistep methods exist.
- AB has to be started with the same-order selfstarting method.

Generally, multistep methods are good for stiff equations (see homework)

Stiffness has no "exact" definition, but it is when there are terms in equations ~~that~~ which can lead to rapid variation of the solution. HD eq-s are stiff.