High-order time integration.

Heuns method:

IVP: $\frac{dy}{dt} = f(t, y(t))$; $y(t_0) = y_0$; find $y(t_1)$

$\Delta t = t_1 - t_0$

Fundamental theorem of Calculus: $y(t_1) = y(t_0) + \int_{t_0}^{t_1} \frac{dy}{dt}\, dt =$

$$= y(t_0) + \int_{t_0}^{t_1} f(t, y(t))\, dt.$$

Now, numerical integration can be used to estimate the integral.

Rectangular integration: ① $y(t_1) = y(t_0) + \Delta t \cdot f(t_0, y(t_0))$ – Euler ~~rate~~ method

Trapezoidal integration: ② $y(t_1) = y(t_0) + \frac{\Delta t}{2}\left(f(t_0, y(t_0)) + \underbrace{f(t_1, y(t_1))}_{\text{Not known.}}\right)$

Use ① Euler method to estimate:

$$y(t_1) = y(t_0) + \frac{\Delta t}{2}\left(f(t_0, y(t_0)) + \underbrace{f(t_1, y(t_0 + \Delta t \cdot f(t_0, y(t_0))))}_{\text{Euler's step, predictor.}}\right)$$

RK2 — Heun's method; improved Euler method. <u>Second order.</u>

Let's take a more general look at the IVP.

$$\frac{dy}{dt} = f(t, y) \qquad (*)$$

Taylor series y on t:

$$y(t+\Delta t) = y(t) + \Delta t \cdot \left.\frac{dy}{dt}\right|_t + \frac{\Delta t^2}{2}\frac{d^2y}{dt^2} + O(h^3).$$

from $(*)$ — from differentiating $(*)$

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(f(t,y)\right) = \frac{df}{dt} + \frac{df}{dy}\frac{dy}{dt} =$$

$$= \frac{df}{dt} + \frac{df}{dy} \cdot f(t,y)$$

Substitute back:

$$y(t+\Delta t) = y(t) + \Delta t \, f(t,y) + \frac{\Delta t^2}{2}\left[\frac{df}{dt} + \frac{df}{dy} f(t,y)\right] =$$

$$= y(t) + \frac{\Delta t}{2} f(t,y) + \frac{\Delta t}{2}\left[f(t,y) + \Delta t \frac{df}{dt} + \underbrace{\Delta t \, f(t,y)}\frac{df}{dy}\right]$$

Use multivariate Taylor expansion for f:

$$f(t+\Delta t, y+\Delta y) = f(t,y) + \Delta t \frac{df(t,y)}{dt} + \Delta y \frac{df(t,y)}{dy} + O(h^2)$$

So, the bracket [ ] is equal to $f(t+\Delta t, y + \Delta t \cdot f(t,y))$

and

$$y(t+\Delta t) = y(t) + \frac{\Delta t}{2} f(t,y) + \frac{\Delta t}{2} f(t+\Delta t, y+\Delta t \cdot f(t,y)) + O(h^3)$$

Or,

$$y_{n+1} = y_n + \Delta t\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)$$
$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + \Delta t, y_n + \Delta t \cdot k_1) \qquad O(\Delta t^3)$$

Same, as above Heun's method; different formulation.

k_1 and k_2 – stages of RK2.

Generally, R-K methods are given by:

$$y_{n+1} = y_n + \Delta t \sum_{j=1}^{\nu} b_j k_j$$

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + c_2 \Delta t, y_n + a_{21} \Delta t k_1)$$
$$k_3 = f(t_n + c_3 \Delta t, y_n + \Delta t(a_{31} k_1 + a_{32} k_2))$$
$$k_\nu = f\left(t_n + c_\nu \Delta t, y_n + \Delta t\left(\sum_{j=1}^{\nu-1} a_{\nu j} k_j\right)\right)$$

- Derivation of coefficients is very messy.
- Maple dies at 6^{th} order (I haven't checked...)
- For higher orders, R-K methods are inefficient because ~~...~~ achievable precision order is less than the number of stages per time step.
- Precise data is still not known for higher orders (>6)

Still. RK4 is the most popular time integration method. Developed in 1900-th.

RK4 (classical.):

$$y_{n+1} = y_n + \Delta t \left[\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \right]$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_1\right)$$

$$k_3 = f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} k_2\right)$$

$$k_4 = f\left(t_n + \Delta t, y_n + \Delta t\, k_3\right).$$