High order finite difference schemes.
Spatial discretisation.

1). CD4:

$$
\begin{array}{l|l}
a\times & f(x+h) = f(x) + f'(x)h + \dfrac{f''(x)h^2}{2!} + \dfrac{f'''(x)h^3}{3!} + O(h^4) \\[2ex]
b\times & f(x-h) = f(x) - f'(x)h + \dfrac{f''(x)h^2}{2!} - \dfrac{f'''(x)h^3}{3!} + O(h^4) \\[2ex]
c\times & f(x+2h) = f(x) + 2f'(x)h + \dfrac{4f''(x)h^2}{2!} + \dfrac{8f'''(x)h^3}{3!} + O(h^4) \\[2ex]
d\times & f(x-2h) = f(x) - 2f'(x)h + \dfrac{4f''(x)h^2}{2!} - \dfrac{8f'''(x)h^3}{3!} + O(h^4)
\end{array}
$$

We want: $f'(x)$.

So,

$$a \cdot f(x+h) + b \cdot f(x-h) + c f(x+2h) + d f(x-2h) =$$

$$= (a+b+c+d) f(x) + \underline{(a-b+2c-2d) f'(x) h} +$$

$$+ (a+b+4c+4d) f''(x) h^2 + (a-b+8c-8d) f'''(x) h^3 + O(h^4)$$

So,

$$
\begin{cases}
a+b+c+d = 0 \\
a-b+2c-2d = 1 \\
a+b+4c+4d = 0 \\
a-b+8c-8d = 0
\end{cases}
\Rightarrow
$$

Shuffling:

$$\to \begin{cases} 2a+3c-d=1 & \xrightarrow{\times 2} \quad 4a+6c-2d=2 \\ 2a+6c+2d=1 & \xrightarrow{\times 2} \quad 4a+12c+4d=2 \\ 2a+12c-4d=1 \end{cases}$$

$$\downarrow$$

$$\begin{cases} 6a+12c=3 \\ 6a+24c=2 \end{cases} \qquad a=\frac{3-12c}{6}$$

$$3-12c+24c=2$$

$$12c=-1$$

$$c=-\frac{1}{12}$$

$$a=\frac{1}{2}-2\left(-\frac{1}{12}\right)=\frac{1}{2}+\frac{2}{12}=\frac{8}{12}\ a$$

Note, system is symm. (a,b), (c,d) so $a=-b$, $c=-d$.

$$d=2a+3c-1=$$

$$=2\cdot\frac{8}{12}+3\left(-\frac{1}{12}\right)-1=\frac{16}{12}-\frac{3}{12}-\frac{12}{12}=\frac{1}{12}=d$$

$$a+b+c+d=0.$$

$$b=-d-a-c=-\cancel{\frac{1}{12}}-\frac{8}{12}+\cancel{\frac{1}{12}}=-\frac{8}{12}=b$$

(don't forget h!)

$$f'(x)=\frac{8}{12h}f(x+h)-\frac{8}{12h}f(x-h)-\frac{1}{12h}f(x+2h)+\frac{1}{12h}f(x-2h)=$$

$$f'=\frac{1}{12h}\Big(f(x-2h)-8f(x-h)+8f(x+h)-f(x+2h)\Big)=$$

$$=\frac{1}{12h}\left(f_{i-2}-8f_{i-1}+8f_{i+1}-f_{i+2}\right).$$

Same way, any other derivative can be approximated. CD4 is unstable, as central scheme.

Higher order precision involves larger stencil.
4-th order – 5 points.
6-th order – 7 points etc.

Stencil size grows with the order of precision. Boundaries become more and more complex, when no information is provided about the values outside the domain.

Compact schemes:

Def. $u_{i+m} = u(x_i + \Delta x \cdot m)$ — derivative.

Taylor series: $u_{i+m} = \sum_{n=0}^{\infty} \frac{(m\Delta x)^n}{n!} u^{(n)}$

Construct: $u_{i+m} \pm u_{i-m} = \sum_{n=0}^{\infty} \left(1 \pm (-1)^n\right) \frac{(m\Delta x)^n}{n!} u^{(n)}$

or,
$$\frac{u_{i+m} + u_{i-m}}{2} = \sum_{n=0,2,4}^{\infty} \frac{(m\Delta x)^n}{n!} u^{(n+1)}$$

and
$$\frac{u_{i+m} - u_{i-m}}{2} = \sum \frac{(m\Delta x)^{n+1}}{(n+1)!} u^{(n+1)}$$

Also, $u \to u'$ (can be differentiated, same shape).

so,
$$\frac{u^{(1)}_{i+m} + u^{(1)}_{i-m}}{2} = \sum_{n=0,2,4}^{\infty} \frac{(m\Delta x)^n}{n!} u^{(n+1)}$$

Lele, JCP, 103, 16, 1992
Shukla & Zhong, JCP, 404, 204, 2005

Consider centered expansion of the form

$$\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} = a_1 \frac{u_{i+1} - u_{i-1}}{2\Delta x} + a_2 \frac{u_{i+2} - u_{i-2}}{4\Delta x} + \frac{u_{i+3} - u_{i-3}}{6\Delta x}$$

Substitute previously found relations:

$$u'_i + 2\alpha \frac{u'_{i-1} + u'_{i+1}}{2} = u'_i + 2\alpha \sum_{n=0,2,4}^{\infty} \frac{\Delta x^n}{n!} u^{(n+1)} =$$

$$= \frac{a_1}{\Delta x} \frac{u_{i+1} - u_{i-1}}{2} + \frac{a_2}{2\Delta x} \frac{u_{i+2} - u_{i-2}}{2} + \frac{a_3}{3\Delta x} \frac{u_{i+3} - u_{i-3}}{2} =$$

$$= \frac{a_1}{\cancel{\Delta x}} \sum \frac{\Delta x^{\cancel{n+1}}}{(n+1)!} u^{(n+1)} + \frac{a_2}{\cancel{2\Delta x}} \sum \frac{(2\Delta x)^{\cancel{n+1}}}{(n+1)!} u^{(n+1)} + \frac{a_3}{\cancel{3\Delta x}} \sum \frac{(3\Delta x)^{\cancel{n+1}}}{(n+1)!} u^{(n+1)} =$$

$$= \sum_{n=0,2,4}^{\infty} \frac{1}{(n+1)!} \Delta x^n u^{(n+1)} (a_1 + 2^n a_2 + 3^n a_3) = u'_i + 2\alpha \sum \frac{\Delta x^n (n+1)}{(n+1)!} u^{(n+1)}$$

$$u'_i + \sum_{n=0}^{\infty} \frac{\Delta x^n}{(n+1)!} u^{(n+1)} 2\alpha (n+1) - \frac{\Delta x^n}{(n+1)!} u^{(n+1)} (a_1 + 2^n a_2 + 3^n a_3) =$$

$$= \sum_{n=0,2,4}^{\infty} \frac{\Delta x^n}{(n+1)!} u^{n+1} \left(2\alpha(n+1) - (a_1 + 2^n a_2 + 3^n a_3)\right) = u'_i$$

$$(2\alpha + \delta_{0n})$$

or

$$\sum_{n=0,2,4\ldots}^{\infty} \frac{\Delta x^n}{(n+1)!} u^{(n+1)} \left(\left(2\alpha + \delta_{0n}\right)(n+1) - \left(a_1 + 2^n a_2 + 3^n a_3\right) \right) = 0.$$

$$n=0: \quad u^{(1)} \left((2\alpha+1)\cdot 1 - (a_1 + a_2 + a_3) \right) = 0.$$

$$\underline{2\alpha + 1 = a_1 + a_2 + a_3}$$

$$n = 2, 4 \ldots N \quad 2\alpha(n+1) = a_1 + 2^n a_2 + 3^n a_3$$

(System of eq-s is already here)

Now, if we truncate our series at some N, leading truncation term will be "$N+2$":

$$\frac{-\left(a_1 + 2^{N+2} a_2 + 3^{N+2} a_3\right) + 2\alpha(N+3)}{(N+3)!} \underline{\underline{\Delta x^{N+2}}} \cdot \underline{\underline{u^{(N+3)}}}$$

Progressively better approximations can be derived. We have 4 parameters (α, a_1, a_2, a_3), need 4 eq-s max.

① Take $a_2 = a_3 = 0$, $n=0$, $n=2$

$$\begin{cases} 2\alpha + 1 = a_1 \\ 2\alpha \cdot 3 = a_1 \end{cases}; \quad 2\alpha + 1 = 6\alpha \quad \boxed{\alpha = \frac{1}{4}} \quad \boxed{a_1 = \frac{2}{4} 3 = \frac{3}{2}}$$

$$\frac{1}{4} u'_{i-1} + u'_i + \frac{1}{4} u'_{i+1} = \frac{3}{2} \frac{u_{i+1} - u_{i-1}}{2\Delta x}.$$

Truncation at $N=2$ => $\boxed{\text{4th order scheme!}}$

If we set $a_3 = 0$ only, a family of numerical methods can be developed:

$$\begin{cases} 2\alpha + 1 = a_1 + a_2 \\ 6\alpha = a_1 + 4a_2 \end{cases} \qquad a_1 = \frac{2}{3}(\alpha + 2) \qquad a_2 = \frac{1}{3}(4\alpha - 1)$$

Note, $\alpha = 0$ reverts to CD4

Now, we substitute a_1 & a_2 to the truncation term and require precision order:

$$N = 2, \quad TE = \frac{-\left(\frac{2}{3}(\alpha+2) + 2^4\frac{1}{3}(4\alpha - 1)\right) + 2\alpha \cdot 5}{5!} \Delta x^4 u^{(5)} + O(\Delta x^6)$$

We require $= 0$ to have $\emptyset$ in front of Δx^4, therefore our scheme will be 6-th order.

$$-\left(\frac{2}{3}(\alpha+2) + 16 \cdot \frac{1}{3}(4\alpha - 1)\right) + 10\alpha = 0 \qquad \times 3$$

$$-\left(2(\alpha+2) + 16(4\alpha-1)\right) + 30\alpha = 0$$

$$-\left(2\alpha + 4 + 64\alpha - 16\right) + 30\alpha = 0.$$

$$-2\alpha - 4 - 64\alpha + 16 + 30\alpha = 0$$

$$-36\alpha + 12 = 0; \qquad \alpha = \frac{1}{3}; \ a_1 = \frac{14}{9}; \ a_2 = \frac{1}{9}$$

Scheme:

$$\frac{1}{3}u'_{i-1} + u'_i + \frac{1}{3}u'_{i+1} = \frac{14}{9}\frac{u_{i+1} - u_{i-1}}{2\Delta x} + \frac{1}{9}\frac{u_{i+1} - u_{i-1}}{4\Delta x}$$

Note, implicit... But not as bad as full implicit schemes, only 1D needed, so, small.

6th and higher order schemes can be developed.