Lax-Wendroff scheme:

L-F scheme is 1st order in time.
So $v\Delta t \ll \Delta x$ to achieve accuracy.

Natural step will be:

$$\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta x} + O(\Delta t^2)$$

This leads (together with CD2 for space) to the Leapfrog scheme:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0 \Rightarrow$$

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + v\,\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x} = 0 \; . \Rightarrow$$

$$u_j^{n+1} = u_j^{n-1} - \frac{\Delta t v}{\Delta x}\left(u_{j+1}^{n} - u_{j-1}^{n}\right) + O(\Delta x^2)$$

Stability (von Neumann):

$$S(k) = -\frac{i\Delta t v}{\Delta x}\sin\theta \pm \sqrt{1 - \frac{\Delta t v^2}{\Delta x^2}\sin^2\theta}$$

$$|S(k)|^2 = \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2\theta + \left(1 - \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2\theta\right) = 1$$

for $\left|\frac{v\Delta t}{\Delta x}\right| \leq 1$

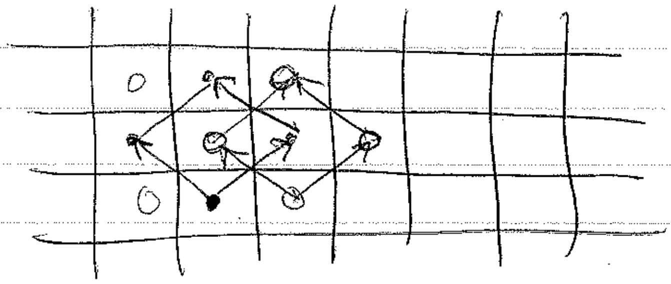
So,

if CFL ≤ 1, stable, and no dissipation.
(exponent does not change)

Leapfrog:

– two-level scheme, which makes it more difficult to use.

– Major disadvantage:



Even and odd points are completely decoupled.
Can be made better by adding (again!) a dissipative term with small coefficient.

Lax-Wendroff:

combination of Leapfrog and Lax-Friedrichs:

$\boxed{1.}$ Make L-F half-step: $\left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \text{ generally}\right)$

$$u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{\Delta t}{2\Delta x}\left(f_{j+1}^n - f_j^n\right)$$

and $+O(\Delta x^2)$

$$u_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(u_j^n + u_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f_j^n - f_{j-1}^n\right)$$

Note, timestep is $\Delta t/2$!

2 As, in general, $f = f(u)$, we need to evaluate fluxes f from the half-step values $u^{n+\frac{1}{2}}$.

3 A Leapfrog half-timestep:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(f_{j+1/2}^{n+1/2} - f_{j-1/2}^{n+1/2}\right) + O(\Delta x^2)$$

For advection equation $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$, reduces to your homework:

$$u_{j\pm 1/2}^{n+1/2} = \frac{1}{2}\left(u_j^n + u_{j\pm1}^n\right) \mp \frac{v\Delta t}{2\Delta x}\left(u_{j\pm1}^n - u_j^n\right),$$

$$u_j^{n+1} = u_j^n - \frac{v\Delta t}{2\Delta x}\left(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}\right) =$$

$$= u_j^n - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{v^2\Delta t^2}{2\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Scheme: