Lax-Friedrichs scheme:

take FT spatially centred scheme

$$u_j^{n+1} = u_j^n - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

Replace u_j^n with the average around j-cell:

$$u_j^n \to \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)$$

This turns into

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

von Neumann stability analysis will show

$$|S(k)|^2 = 1 - \sin^2\theta\,(1-R^2),$$

leads to the same CFL — $R^2 < 1$

$$\left|\frac{v\Delta t}{\Delta x}\right| < 1, \quad \Delta t \leq \frac{\Delta x}{v}$$

— Similar trick with the diffusion term:
The scheme can be rewritten as;

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = -v\left(\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}\right)+\frac{1}{2}\left(\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta t}\right)$$

which is exactly the finite-difference representation of

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x} = \underbrace{\frac{1}{2}\frac{\Delta x^2}{\Delta t}\frac{\partial^2 u}{\partial x^2}}_{\text{Diffusion term again.}}$$

$$k = \frac{1}{2}\frac{\Delta x^2}{\Delta t}$$

(Interestingly, it can be reduced without drama)

Objection: we are solving different equation here

However,

(1) $\frac{\partial^2 u}{\partial x^2}$ in FD is $O(h^2)$,

multiplied by $\frac{\Delta x^2}{\Delta t} = O(h)$, gives $\boxed{O(h^3)}$.

On the other hand, LHS is only $O(h)$ (time deriv.) and $O(h^2)$ (spat. der.).

So, the diffusion term goes to 0 with $h \to 0$ faster, than other terms.

Therefore, L-F scheme converges to the correct solution.

(2) The length scales we are interested in are large, meaning

$\theta = k\Delta x$ is small. So, $\sin^2\theta$ is small, and $S(k)$ is very close to 1 at large scales.

This corresponds to the modes staying at the same amplitude over time.

(3) In the case of Central scheme, ~~the~~ small spatial scales will blow up, killing all the solution.

It is preferred to have stable but inaccurate scheme than unstable.

1D Lax-Wendroff scheme, multistep methods — next.

Note, h^1-order of accuracy in previous schemes comes from temporal discretisation.

Few things to note:

1) CSHIFT for some reason works slower than DO loop. Check it with python?

2) The problem allows changes in the initial conditions. Use different perturbation ampl./period to get different pictures. Shape too.

3) The code you have allows a lot to model, simple and fast. + supersonic.

Prev. details of derivation!
Why diffusion?

$$\frac{q^{n+1}-q^{n}}{\Delta t}+v\boxed{\frac{q_i-q_{i-1}}{\Delta x}}=0 \Leftrightarrow \frac{\partial q}{\partial t}+v\frac{\partial q}{\partial x}=0$$

$$\frac{q_i-q_{i-1}}{\Delta x}=\frac{q_{i+1}-q_{i-1}}{2\Delta x}-\frac{\Delta x}{2\Delta x^2}\left(q_{i+1}-2q_i+q_{i-1}\right)=$$

$$=\cancel{\frac{q_{i+1}}{2\Delta x}}-\frac{q_{i-1}}{2\Delta x}-\cancel{\frac{q_{i+1}}{2\Delta x}}+\cancel{\frac{2q_i}{2\Delta x}}-\frac{q_{i-1}}{2\Delta x}=$$

$$=\frac{q_i}{\Delta x}-\frac{2q_{i-1}}{2\Delta x}=\frac{q_i-q_{i-1}}{\Delta x}.$$

But $\frac{\Delta x}{2\Delta x^2}\left(q_{i+1}-2q_i+q_{i-1}\right)$ is $\frac{\Delta x}{2}\frac{\partial^2 q}{\partial x^2}$

So, if substituted back to the equation

$$\underbrace{\frac{\partial f}{\partial t}}_{\text{upwind}} + v \underbrace{\frac{\partial f}{\partial x}}_{\text{upwind}} = 0$$

is equivalent to

$$\underbrace{\frac{\partial f}{\partial t}}_{\text{upwind}} + v \underbrace{\frac{\partial f}{\partial x}}_{\text{central}} = \underbrace{\frac{v\Delta x}{2}}_{\text{diffusion coefficient.}} \underbrace{\frac{\partial^2 f}{\partial x^2}}_{\text{2nd derivative central.}}$$

For Lax-Friedrichs scheme:

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - v\Delta t \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

Subtract u_j^n, divide by Δt

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n}{2\Delta t} + \frac{u_{j-1}^n}{2\Delta t} - v \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{u_j^n}{\Delta t} =$$

$$= -v \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta t} =$$

$$= \underbrace{-v \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}}_{\text{central}} + \underbrace{\frac{\Delta x^2}{2\Delta t}}_{\text{diffusion coefficient.}} \underbrace{\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}}_{\frac{\partial^2 u}{\partial x^2}}$$