Advection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad u = u(x,t)$$

gen. sol. $u(x,t) = f(x-at)$

$a>0 \rightarrow$ right-travelling wave

Temporal discretisation:

$$\frac{u^{n+1}-u^n}{\Delta t} + a\frac{\partial u^n}{\partial x} = 0.$$ Forward Euler (explicit).

$$\frac{u^{n+1}-u^n}{\Delta t} + a\frac{\partial u^{n+1}}{\partial x} = 0$$ Backward Euler (implicit).

$$\frac{u^{n+1}-u^n}{\Delta t} + \frac{a}{2}\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial u^n}{\partial x}\right) = 0$$ Crank-Nicolson (implicit).

There are better explicit schemes (e.g. R-K).
Runge-Kutta.

Von Neumann stability
introduced later.

Use Forward Euler scheme for time. ②

1.

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + a \underbrace{\frac{v_i^n - v_{i-1}^n}{\Delta x}}_{\text{upwind derivative}} = 0.$$

von-Neumann stability: LOCAL :

1) does not take into account boundaries
2) coefficients of FDE's vary slowly.

$$v_i^n = u_i^n - e_i^n$$

numerical solution, exact solution, actual error at x_i, t_n

Substitute to the equation:

$$\underbrace{\left[\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x}\right]}_{\text{Truncation error}} - \underbrace{\left[\frac{e_i^{n+1} - e_i^n}{\Delta t} + a\frac{e_i^n - e_{i-1}^n}{\Delta x}\right]}_{\text{Propagation equation (time-dependent) for actual error.}} = 0.$$

Method is numerically stable if the actual error is bounded: $e_i^n \leq \varepsilon$ with $n \to \infty$

unstable if grows without bound (numerical instability).

Assume truncation error is zero.

$$\frac{e_j^{n+1} - e_j^n}{\Delta t} + a\,\frac{e_j^n - e_{j-1}^n}{\Delta x} = 0$$

e_j^n has to be bounded for stability.

Decompose e in Fourier modes:

$$e_j^n = \hat{e}^n \exp(ikx_j)$$

here:

$\hat{e}^n$ is amplitude of the wave mode at time n (complex)

k is wavenumber.

$$k = \frac{2\pi}{\lambda}; \quad x_j = j\Delta x; \quad \theta = k\Delta x$$

$$e_j^n = \hat{e}^n \exp(ikj\Delta x)$$

$$e_j^n = \hat{e}^n \exp(ij\theta)$$

We are interested in behaviour of e_j^n on time, in particular $\tilde{e}^n$ evolution with each time step.

If $\tilde{e}^n$ is bounded, the method is numerically stable. (for all θ)

Substitute $e_j^n = \hat{e}^n \exp(ij\theta)$ into error propagation equation;

$$\frac{\tilde{e}^{n+1}\exp(ij\theta) - \tilde{e}^{n}\exp(ij\theta)}{\Delta t} + a\,\frac{\tilde{e}^{n}\exp(ij\theta) - \tilde{e}^{n}\exp(i(j-1)\theta)}{\Delta x} = 0$$

$$\tilde{e}^{n+1}\exp(ij\theta) = \tilde{e}^{n}\exp(ij\theta) - \frac{a\Delta t}{\Delta x}\tilde{e}^{n}\left(\exp(ij\theta) - \exp(i(j-1)\theta)\right)$$

divide by $\exp(ij\theta)$

$$\tilde{e}^{n+1} = \tilde{e}^{n} - \frac{a\Delta t}{\Delta x}\tilde{e}^{n}\left(1 - \exp(-i\theta)\right)$$

$$\frac{\tilde{e}^{n+1}}{\tilde{e}^{n}} = 1 - \frac{a\Delta t}{\Delta x}\left(1 - \exp(-i\theta)\right) =$$

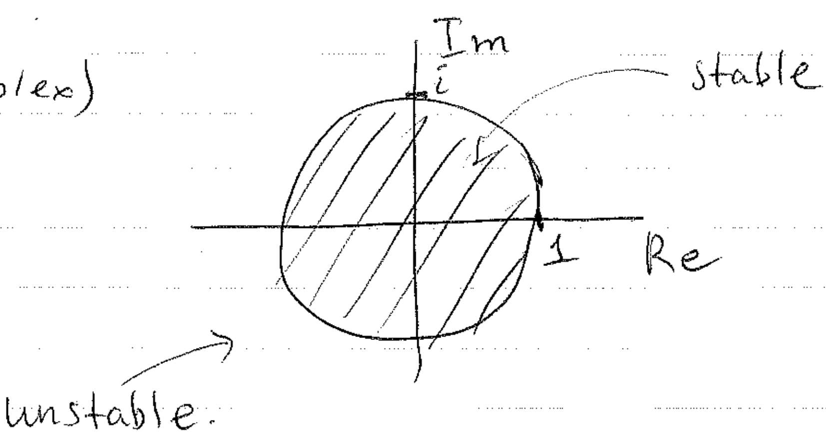
$\exp(i\theta) = \cos\theta - i\sin\theta$

$$\boxed{\frac{\tilde{e}^{n+1}}{\tilde{e}^{n}} = 1 - \frac{a\Delta t}{\Delta x} + \frac{a\Delta t}{\Delta x}\cos\theta - i\,\frac{a\Delta t}{\Delta x}\sin\theta}$$

Def: $S(k) = \dfrac{\tilde{e}^{n+1}}{\tilde{e}^{n}}$ for 2-level FD method.

$\max\limits_{k} |S(k)| \leq 1$ – von Neumann stability condition.

($S(k)$ – complex)



$$S(k) = 1 - \frac{a\Delta t}{\Delta x} + \frac{a\Delta t}{\Delta x}\cos\theta - i\,\frac{a\Delta t}{\Delta x}\sin\theta.$$

$$R = \frac{a\Delta t}{\Delta x}$$

$$S(k) = 1 - R + R\cos\theta - iR\sin\theta$$

$$C(\theta) = |S(k)|^2 = 1 - 2R(1-\cos\theta) + R^2(1-\cos\theta)^2 + R^2\sin^2\theta =$$

$$= 1 - 2R(1-\cos\theta) + R^2(1 - 2\cos\theta + \cos^2\theta) + R^2\sin^2\theta =$$

$$= 1 - 2R(1-\cos\theta) + 2R^2 - 2R^2\cos\theta =$$

$$= 1 - 2R(1-\cos\theta) + 2R^2(1-\cos\theta) =$$

$$= 1 - (1-\cos\theta)(2R - 2R^2) = \underline{1 - (1-\cos\theta)\,2R(1-R)}$$

Determine extrema of $C(\theta)$

$$\frac{dC(\theta)}{d\theta} = -2R(1-R)\sin\theta = 0.$$

$$\theta = 0, \pm\pi, \pm 2\pi \ldots \Rightarrow \underline{\cos\theta = \pm 1.}$$

$$\max|S(k)|^2 = \max\left|1 - 2R(1-R)(1 \pm \overset{(\cos\theta)}{1})\right| =$$

$$= \max\left|1, |1 - 4R(1-R)|\right| =$$

$$= \max\left|\underset{\text{good.}}{\underline{1}}, |1-2R|^2\right).$$

$$1 - 4R(1-R) = 1 - 4R + 4R^2 = (1-2R)^2$$

$$\max|S(k)| = \max\left(1, |1-2R|\right)$$

$$|1-2R| \leq 1 \Rightarrow R \leq 1, \; R \geq 0.$$

$$0 \leq R \leq 1$$

$$0 \leq \frac{a\Delta t}{\Delta x} \leq 1 \qquad \times \frac{\Delta x}{a}$$

$$0 \leq \Delta t \leq \frac{\Delta x}{a}$$

$\Delta t \leq R \frac{\Delta x}{a}$ — stable if true.

Courant - Friedrichs - Lewy condition.
The scheme is 'conditionally stable'!

Note, for upwind scheme $\boxed{a > 0}$.

Otherwise, unconditionally unstable!

(So, pretty much useless scheme, if used in CFD with arbitrary flow speed).
Linked with information propagation direction and asymmetry of the scheme).

There are unc. unstable, unc. stable (Crank-Nicolson) cond. stable (subject to CFL) schemes.

Similar derivation for forward-Euler spatially centered scheme leads to (can be exam prob.)

$$|S(k)| = \sqrt{1+R^2\sin^2\theta} \geq 1$$

$$\max_k |S(k)| > 1 \text{ for any } R.$$

Therefore, FTCD scheme is unconditionally unstable.

Stability & diffusion.

Consider an upwind scheme.

$$\frac{V_i - V_{i-1}}{\Delta x} = \frac{V_{i+1} - V_{i-1}}{2\Delta x} - \frac{\Delta x}{2\Delta x^2}\left(V_{i+1} - 2V_i + V_{i-1}\right)$$

If substituted back to the equation, leads to

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} \underbrace{- \frac{\Delta x u}{2}\frac{\partial^2 v}{\partial x^2}}_{\text{Diffusion term, if } u>0.} = 0.$$

Diffusion coefficient $D = \frac{\Delta x u}{2}$;

$\Delta x \to 0$ diffusion vanishes.

What happens? Diffusion makes the scheme stable.

Concept of "artificial viscosity", which stabilises solution.

$\boxed{\text{Some diffusivity is inevitable!}}$