Consistensy;

a numerical method $A^h V^h = F^h$ is **consistent** with PDE $Lu = f$

if $\lim_{h \to 0} T^h = 0$;

order of consistensy q if $T = O(h^q)$

Relation between T^h & E^h:

$$T^h = A^h U^h - F^h$$

Subtract: $A^h V^h = F^h$; $\quad 0 = A^h V^h - F^h$

$$T^h = A^h (U^h - V^h) \quad ; \quad U^h - V^h = E^h$$

$$\boxed{T^h = A^h E^h}$$

$$\left\| (A^h)^{-1} \left[A^h E^h = T^h \right] \right\|_p =$$

$$\boxed{\begin{array}{l} p\text{-norm:} \\ \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \\ p=1 \text{ "Taxicab"} \\ p=2 \text{ Euclidean} \end{array}}$$

$$= \|E^h\|_p = \|(A^h)^{-1}\|_p \, \|T^h\|_p$$

If we know that a numerical method is consistent, i.e. $T^h = O(h)$ atleast, then

if exists c sothat $\|(A^h)^{-1}\|_p \le c$ $\quad (c_s \neq c_s(h))$

$A^h V^h = F^h$ is stable

Lax convergence theorem:

$$\|E^h\|_p = \|(A^h)^{-1}\|_p \, \|T^h\|_p$$

$$\|E^h\|_p = O(h^q) \quad \Longleftarrow \quad \leq C_s \quad \longleftarrow + \longleftarrow \quad O(h^q)$$

If the metod is consistent with order q in the p-norm and stable in the p-norm, then the method is convergent with the order q.

(Fundamental theorem of Finite Difference methods)

Note: E^h converges with the same order as T^h. T^h can be used to obtain order of convergence.

Example:

$$T_i^h = \frac{1}{12} h^2 \frac{\partial^4 u}{\partial x^4} + O(h^3) = O(h^2).$$

can be shown! $T_i = \frac{1}{12} h^2 u^{(4)} \gamma(x_i)$

(1) consistent with the order 2 with our problem.

Assume $C_T = \max_{x \in [0,1]} |u^{(4)}(x)|$

$$T_i \le \frac{1}{12} h^2 C_T$$

2-norm

$$\|T^h\|_2 = \sqrt{h} \sqrt{\sum_{i=1}^{m} (T_i)^2} \le \sqrt{h} \sqrt{m \left(\frac{1}{12} h^2 C_T\right)^2} \le$$

$$\le \sqrt{h} \sqrt{(m+1)\left(\frac{1}{12} h^2 C_T\right)^2}$$

$h = \frac{1}{m+1}$ (from BVP definition)

$$\Downarrow$$

$$\|T^h\|_2 \le \frac{1}{12} h^2 C_T$$

$$\Downarrow$$

$$\|T^h\|_2 = O(h^2)$$

/ consistent.

2. Stability;

1) A^h is symmetric $\Rightarrow (A^h)^{-1}$ is symmetric too:

$$A^h = \cancel{\ } (A^h)^T \Rightarrow (A^h)^{-1} = ((A^h)^{-1})^T.$$

Spectral radius of A, $\rho(A) = \max\limits_{1 \le i \le m} |\lambda_i|$,

where λ_i – eigenvalues of A given by $\det(A - \lambda I) = 0$

$$(A - \lambda I)\vec{v} = 0$$

It can be shown that if $A = A^T$, $\|A\|_2 = \rho_A$

$\left(2\text{-norm } \|A\|_2 = \sqrt{\rho(AA^T)} = \sqrt{\rho(A^TA)} \right)$

So, $\|(A^h)^{-1}\|_2 = \rho((A^h)^{-1})$.

2). If A^h is invertible and has eigenvalues $\lambda_1 \dots \lambda_m$, $(A^h)^{-1}$ has eigenvalues $\frac{1}{\lambda_1} \dots \frac{1}{\lambda_m}$

$$A\vec{v} = \lambda\vec{v} \ \Big| \text{ mult. by } \frac{A^{-1}}{\lambda} \Rightarrow \frac{1}{\lambda}\vec{v} = A^{-1}\vec{v},$$

which implies that $\frac{1}{\lambda}$ are eigenvalues.

Using 1) and 2),

$$\|(A^h)^{-1}\|_2 = \rho((A^h)^{-1}) = \max_{1 \le i \le m} \left|\frac{1}{\lambda_i}\right| = \left(\min_{1 \le i \le m} |\lambda_i|\right)^{-1}.$$

Need eigenvalues for A.

Sturm-Liouville problem.
(discrete)

* Note: solution is ~~an~~ sum of eigenfunctions; solution depends on how they behave

Need eigenvalues for A.

$$A^{\lambda} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & & \dots & 1 & -2 \end{pmatrix}$$

$$Au = \lambda u$$

one of the equations $u_{j-1} - 2u_j + u_{j+1} = \lambda u_j$

It can be shown that for A^h eigenvalues are

$$\lambda_k = \frac{2}{h^2}\left(\cos(k\pi h) - 1\right), \quad k = 1 \dots m$$

$$h = \frac{1}{m+1}$$

λ is smallest when $k=1$ (argument of cos is smallest, $\cos \to 1$)

Expand $\cos(k\pi h)$ around $k = \cancel{1}$

$$\cos(\pi h) = 1 - \frac{1}{2}\pi^2 h^2 + \frac{1}{24}\pi^4 h^4 \cos(\pi \xi)$$
where $\xi \in [0, h]$

Substitute $\cos(\pi h)$ in λ_K :

$$\lambda_1 \approx \frac{2}{h^2}\left(-\frac{1}{2}\pi^2 h^2 + \frac{1}{24}\pi^4 h^4 \cos(\pi\xi)\right) =$$

$$= -\pi^2 + \frac{1}{12}\pi^4 h^2 \cos(\pi \xi)$$

$|\lambda_1| \simeq \pi^2$ independent on h for sufficiently small h.

So

$$\frac{1}{\pi^2} \approx \|(A^h)^{-1}\|_2 = \frac{1}{|\lambda_1|}$$

So the method is stable.

Therefore by Lax convergence, the method A is convergent with order 2 in the 2-norm:

$$\|E^h\|_2 = O(h^2)$$

Note, higher-order methods can be developed.

Few notes:

(1) Sometimes, solving $\mathcal{L}u = \rho$, where $\mathcal{L}$ is Elliptic is complicated.
There are relaxation methods:

$$\mathcal{L}u - \rho = 0 \implies \frac{\partial u}{\partial t} = \mathcal{L}u - \rho$$

u relaxes to equilibrium solution with $t \to \infty$.

(2) Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

can be discretised directly.
But

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{is} \quad \mathcal{L}_1 \mathcal{L}_2 u = 0,$$

and $\mathcal{L}_1 = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$; $\mathcal{L}_2 = \frac{\partial}{\partial t} - v\frac{\partial}{\partial x}$

$$w = \mathcal{L}_2 u \implies$$

$$L_2 u = \omega$$
$$L_1 \omega = 0.$$

Or, in matrix form

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ \omega \end{bmatrix} + \begin{bmatrix} -v & 0 \\ 0 & v \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u \\ \omega \end{bmatrix} = \begin{bmatrix} \omega \\ 0 \end{bmatrix}$$

Which gives a "flux-conservative" equation

$$\frac{\partial \vec{u}}{\partial t} - \frac{\partial \vec{F}(\vec{u})}{\partial x} = 0 \text{ or } \vec{S}$$

(Here, all HD-equations are!)

Prototype:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0.$$

Note, linear!

Let's look at temporal discretisation!

explicit

computes $n+1$ state from $n, n-1, n-2 \ldots$

$\boxed{\text{cheap, but unstable}}$

implicit

computes $n+1$ state from $n+1, n, n-1, n-2$

$\boxed{\text{leads to system of eqs} \Rightarrow \text{expensive, more stable (largest)}}$