NSE show mixed behaviour due to their complexity → they are hyperbolic + parabolic + Elliptic

We have to be able to solve!

Hyperbolic eq: → propagation of information at finite speeds. (~ wave equation)
$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$

Parabolic eq: → diffusion/thermal conduction eq information travels downstream
$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$

Elliptic eq: → Laplace eq., steady state problem; information is 'infinitely fast'!
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Note: unsteady problems are not elliptic

To solve NSE we need to be able to solve all these types of eq-s.

We are going to solve all of that with FINITE DIFFERENCES.

- simple, effective, easy to derive
- limited to block geometry.

There are also Finite Volumes, Finite elements, SPH (Daniel Price) etc etc.

Consider this boundary value problem (BVP)

$$\begin{cases} \Omega = \{x : x \in (0,1)\} & \text{— domain} \\ u(0) = a;\ u(1) = b & \text{— boundary conditions} \\ \frac{d^2u}{dx^2} = f(x) & \text{— eq. to solve} \end{cases}$$

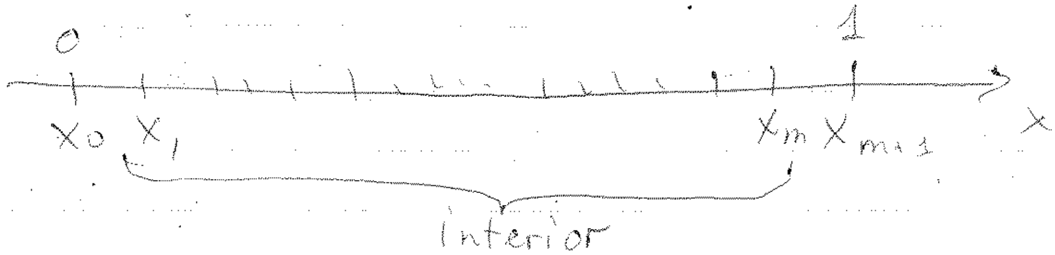
<u>Solution u.</u>

We discretise Ω using $m+2$ points $x = x_0 \dots x_{m+1}$ ($m+1$ intervals)

x_0, x_{m+1} — boundary points (ghost cells)

$x_1 \dots x_m$ — interior points

For all i $\quad x_{i+1} - x_i = \Delta x = h$



Now, we expand $u(x)$ in a Taylor series at $i+1$ and $i-1$ according to:

$$u_{i+1} = u_i + h \cdot \frac{du}{dx}(x_i) + \frac{1}{2} h^2 \frac{d^2u}{dx^2}(x_i) + \frac{1}{6} h^3 \frac{d^3u}{dx^3} + \frac{1}{24} h^4 \frac{d^4u}{dx^4} + O_5$$

$$u_{i-1} = u_i - h \frac{du}{dx}(x_i) + \frac{1}{2} h^2 \frac{d^2u}{dx^2}(x_i) - \frac{1}{6} h^3 \frac{d^3u}{dx^3} + \frac{1}{24} h^4 \frac{d^4u}{dx^4} + O_5$$

Summing them:

$$u_{i+1} + u_{i-1} = 2u_i + h^2 \frac{d^2u}{dx^2}(x_i) + \frac{1}{12} h^4 \frac{d^4u}{dx^4} + O^5 \quad (O(h^5))$$

$$* \quad \boxed{\frac{d^2u}{dx^2}(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}} + \frac{1}{12} h^2 \frac{d^4u}{dx^4} + O(h^3).$$

To the second order, the derivative can be approximated with *.

Now, v_i is approximation to the exact solution u_i.

Using *, we define approx solution v_i for BVP as a unique discrete function v_i, which satisfies

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = f_i, \quad i = 1 \ldots m$$

$$v_0 = a$$

$$v_{m+1} = b.$$

It can be written in matrix form:

$$A^h V^h = \underbrace{F^h_m}$$

$$A^h_{\not{h}} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & 1 & -2\ 1 & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & & 1 & -2 \end{pmatrix}$$ — matrix defines equation.

$$V^h = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix}$$ — solution

$$F^h = \begin{pmatrix} f_1 - \alpha\frac{1}{h^2} \\ f_2 \\ f_3 \\ \vdots \\ f_m - b\frac{1}{h^2} \end{pmatrix}$$ — RHS

Actual error: $E^h = u^h - V^h$

error — exact sol on the grid — approx. solution.

Convergence: we want $E^h \to 0$ as $h \to 0$.

Truncation error: exact solution is plugged into the discretised equation.

$$T^h = A^h u^h - F^h$$

(In our case, $T_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - f_i$,

or $T_i = u'' + \frac{1}{12}h^2 \frac{\partial^4 u}{\partial x^4}(x_i) - f_i + O(h^3)$

u is exact so $\frac{\partial^2 u}{\partial x^2} - f = 0 \Rightarrow T_i = \frac{1}{12}h^2\frac{\partial^4 u}{\partial x^4}(x_i) + O(h^3) = O(h^2)$