Reynolds number / incompressible NS-equations.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0.$$

$$\underbrace{\frac{\partial \rho}{\partial t} + v\cdot\nabla\rho}_{} + \rho\nabla\cdot v = 0.$$

— advective (Lagrangian) derivative.
follows the fluid volume.

$$\frac{D\rho}{Dt} + \rho\nabla\cdot v = 0.$$

Incompressible, when $\frac{D\rho}{Dt} = 0$ or
fluid parcel does not change ρ in its evolution.

$$\frac{D\rho}{Dt} = 0 \Rightarrow \rho\nabla\cdot v = 0 \Rightarrow \underline{\underline{\nabla\cdot v = 0}}.$$

incompressibility

$$\frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v v) + \nabla p = \mu\nabla^2 v + \ldots$$

— Newtonian fluid.
forces.

μ - dynamic viscosity.

$$\rho\frac{\partial v}{\partial t} + \underbrace{v\frac{\partial \rho}{\partial t} + v\nabla\cdot(\rho v)}_{} + \rho(v\cdot\nabla)v + \nabla p = \mu\nabla^2 v$$

continuity eq-n $= 0$.

$$\rho\frac{\partial v}{\partial t} + \rho(v\cdot\nabla)v + \nabla p = \mu\nabla^2 v$$ — still compressible.

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v + \frac{1}{\rho_0}\nabla p = \nu\nabla^2 v,$$

$$\nu = \frac{\mu}{\rho_0}$$

$\rho = \rho_0$ - const — incompressible + homogeneous $\Rightarrow$ different from $\nabla\cdot v = 0$, although implies.

$$\rho \frac{Dv}{Dt} + \nabla p = \mu \nabla^2 v \,; \qquad \frac{Dv}{Dt} + \frac{1}{\rho}\nabla p = \frac{\mu}{\rho}\nabla^2 v$$

Non-dimensionalise:

$$x' = \frac{x}{L}\,; \quad u' = \frac{u}{U}\,; \quad t' = t\frac{U}{L}\,; \quad p' = \frac{p}{\rho U^2}$$

$$x = Lx'\,; \quad u = Uu'\,; \quad t = t'\frac{L}{U}\,; \quad p = p'\cdot\rho U^2$$

$$\frac{D\,Uu'}{D\,t'\frac{L}{U}} + \frac{1}{\rho}\frac{\partial(p'\rho U^2)}{\partial(Lx')} + \frac{\mu}{\rho}\frac{\partial^2 Uu'}{\partial L^2 x'^2} = 0.$$

$$\frac{U^2}{L}\frac{Du'}{Dt'} + \frac{U^2}{L}\frac{\partial p'}{\partial x'} + \frac{\mu}{\rho}\frac{U}{L^2}\frac{\partial^2 u'}{\partial x'^2} = 0.$$

$$\frac{U^2}{L}\left(\frac{Du'}{Dt'} + \frac{\partial p'}{\partial x'} + \underbrace{\frac{1}{UL}\frac{\mu}{\rho}}_{\frac{1}{Re}}\frac{\partial^2 u^2}{\partial x'^2}\right) = 0$$

For incompressible flows, it is good to define

$\nu = \frac{\mu}{\rho}$ — kinematic viscosity

$$\frac{1}{Re} = \frac{\nu}{UL}\,, \qquad Re = \frac{UL}{\nu}$$

Otherwise

$$\frac{1}{Re} = \frac{\mu}{UL\rho}\,, \qquad Re = \frac{UL\rho}{\mu}$$

$Re \approx 14 \div 140 \sim 100$ for Lax-~~F~~riedrichs scheme (Sod test)

Reynolds number – Introduced by Stokes.
popularised by Reynolds.

Many things about it. But, generally,

Low Re – laminar flow.
High Re – turbulent flow.

$Re \sim \frac{1}{\mu}$. The larger the viscosity (diffusivity), the smaller Re, more laminar flow.

In CFD, we need diffusion to stabilise solution. This limits Re range we are able to simulate. To low Re ... $\sim 10^3 - 10^4$ max.

Note, $Re = \frac{uL}{\nu} \sim L^2$

If $L = N dx$, upper bound on $Re \leq N^2$.

Further, Taylor series truncation leads to

$R \leq a N^h$, where h is the order,
a is some constant ~ 1.

The higher the order, the higher ~~Re~~ can be achieved.

DNS (direct Navier-Stokes) vs LES (Large-eddy) simulations.

DNS: Space.

(Kolmogorov turbulence scale, from dimensional analysis:

$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$, where ν is kinematic visc.;
ε is rate of energy dissipation
η – Kolmogorov length.

η is the smallest scale in a turbulent system.)

L - largest scale of our system.
So, our numerical domain has to be:

$Nh > L$, where N - number of grid cells
h - grid cell size.

AND

$h < \eta$, so the Kolmogorov scale can be resolved.

Now, $\varepsilon = \frac{E}{T}$ (rate of energy change per time scale)

$E \propto V^2$; we have no timescale, so replace it with $T = \frac{L}{V}$

$\varepsilon \propto \frac{V^3}{L}$; substitute it to η

$$\eta = \left(\frac{\nu^3 L}{V^3}\right)^{1/4} = \left(\frac{\nu^3 L \cdot L^3}{U^3 L^3}\right)^{1/4} = \left(\frac{L^4}{Re^3}\right)^{1/4} =$$

$\eta = L \cdot \left(\frac{1}{Re}\right)^{3/4}$. Inversely proportional to $Re^{3/4}$

Now, $L = \eta \cdot Re^{3/4}$.

Substitute it to $Nh > L$:

$$Nh > \eta \cdot Re^{3/4}; \quad h \approx \eta;$$

$$N\eta > \eta \cdot Re^{3/4};$$

$$N > Re^{3/4}.$$

In 3d, $N^3 > Re^{9/4} = Re^{2.25}$.

N^3 – amount of grid cells, grows fast with Re.

DNS: Time:

CFL: $C = \frac{V \Delta T}{h} < 1$.

Total simulation time $\tau = \frac{L}{V}$.

$$\Delta T = \frac{Ch}{V}; \quad N = \frac{\tau}{\Delta t} = \frac{L}{\cancel{V}} \frac{\cancel{V}}{Ch} = \frac{L}{Ch}.$$

number of timesteps

$C \approx 1$;
$h \approx \eta$; $\Rightarrow$ $N = \frac{L}{\eta} = Re^{3/4}$

Also growing with Re.

All together simulation
$$N^3_{Space} \cdot N_{time} \approx Re^{2.25} \cdot Re^{0.75} = Re^3$$

Number of operations $\sim Re^3$;

Still prohibitively computationally expensive for most applications, except - turbulence models.

So, large-eddy simulations.

Derived ~~exactly~~ in a similar way :) as we did with NS-equations.
A filter is applied, which has spatial and temporal cutoff scale.

So, any scalar or vector field can be split into a "macroscopic" - filtered, and "microscopic" - - sub-filtered components, i.e.
$$V = \overline{V} + \tilde{V}$$

There will appear tensors of a similar shape to stress tensor in NS-e. A model for them is needed - called subgrid-scale model (SGS) or sub-filter model. Different models exist.