1. We start with the distribution function f defined in 6-d space of coordinates + momenta

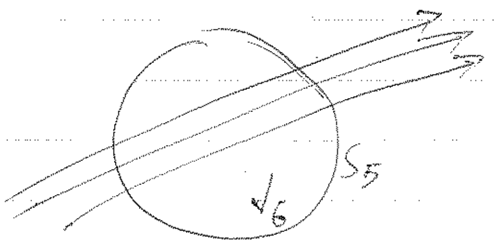
$$f(\vec{r}, \vec{p}, t) = f(r_i, p_i, t)$$

Full differential

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial r_i} dr_i + \frac{\partial f}{\partial p_i} dp_i \, , \quad \text{or}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r_i} \underbrace{\frac{dr_i}{dt}}_{v_i = \frac{p_i}{m}} + \frac{\partial f}{\partial p_i} \underbrace{\frac{dp_i}{dt}}_{F_i = a_i \cdot m}$$

; m – particle mass.



V_6 – arbitrary volume bounded by S_5

We can write Gauss theorem $\left[\int_V (\nabla \cdot \vec{F}) \, dV = \int (\vec{F} \cdot \vec{n}) \, dS \right.$ $\left. \vec{F} \text{ – cont. diff vector field} \right]$

$$\frac{\partial}{\partial t} \int_{V_6} f \, dV_6 + \oint_{S_5} f v_i n_i \, dS_5 = 0$$ – conservation of particle number

$$\frac{\partial}{\partial t} \int_{V_6} f \, dV_6 + \int_{V_6} \frac{\partial}{\partial x_i} (f v_i) \, dV_6 = 0.$$

V_6 – arbitrary, in $\mathbb{R}^n$, $n=6$.

So

$$\boxed{\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i} (f \cdot v_i) = 0} \qquad (1)$$

Splitting (1) in x and v components:

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial x_i}\left(f \cdot \frac{dx_i}{dt}\right) + \frac{\partial}{\partial p_i}\left(f \frac{dp_i}{dt}\right) = 0.$$

$$f \equiv f(x_i, p_i, t)$$

(Liouville equation)

Opening derivatives gives:

$$\frac{\partial}{\partial t} f + \frac{dx_i}{dt}\frac{\partial f}{\partial x_i} + \frac{dp_i}{dt}\frac{\partial f}{\partial p_i} + f \underbrace{\left[\frac{\partial}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial}{\partial p_i}\frac{dp_i}{dt}\right]} = 0$$

Proof that the last bracket is eq to 0:
Assume closed system, write Hamiltonian eqs:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial x_i};$$

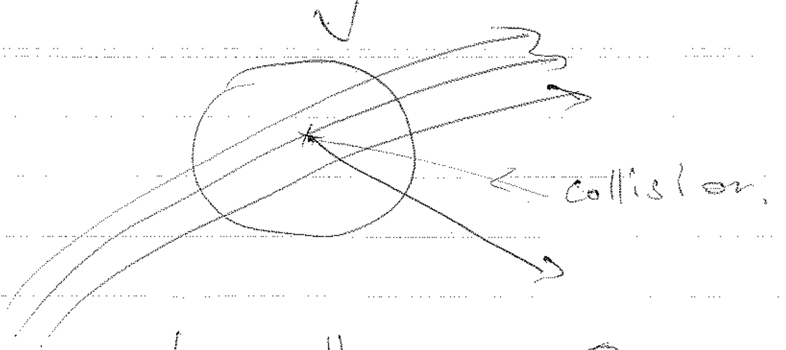
$$\frac{\partial}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial H}{\partial x_i} \equiv 0;$$

Interestingly, the bracket is 6D-divergence of 6D-flow in phase space.

Liouville equation now:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial p_i} = 0.$$

Collisions change:



$$\frac{\partial f}{\partial t}+v_i\frac{\partial f}{\partial x_i}+F_i\frac{\partial f}{\partial p_i}=\left(\frac{\delta f}{\delta t}\right)_{coll}$$ — formally. (*)

(Boltzmann equation)

WE WANT TO GET RID OF VELOCITY PART IN (*).
WE INTEGRATE (*) OVER AN ARBITRARY VOLUME IN V-SPACE $d^3\vec{v}$.

WE ALSO MULTIPLY (*) BY SOME FUNCTION OF $\vec{v}$, $\chi(\vec{v})$

$$\chi = \chi(\vec{v})$$

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \vec{F}\cdot\nabla_p f = \left(\frac{\partial f}{\partial t}\right)_{coll.}$$

$$\Downarrow$$

$$\underbrace{\int \chi \frac{\partial f}{\partial t} d^3\vec{v}}_{\boxed{1}} + \underbrace{\int \chi \vec{v}\cdot\nabla f\, d^3\vec{v}}_{\boxed{2}} + \underbrace{\int \chi F\cdot\nabla_p f\, d^3\vec{v}}_{\boxed{3}} = \underbrace{\int \chi\left(\frac{\partial f}{\partial t}\right)_{coll} d^3\vec{v}}_{\boxed{4}}$$

$\boxed{1}$ $$\int \chi \frac{\partial f}{\partial t} d^3v = \frac{\partial}{\partial t}\int \chi f\, d^3v - \cancel{\int \frac{\partial \chi}{\partial t} f\, d^3v} \to 0. \quad \left(\chi = \chi(\vec{v}) \Rightarrow \frac{\partial \chi}{\partial t} = 0\right)$$

$$\boxed{\int \chi \frac{\partial f}{\partial t} d^3v = \frac{\partial}{\partial t}\int \chi f\, d^3v}$$

$\boxed{2}$ $$\int \chi \vec{v}\cdot\nabla f\, d^3\vec{v} = \nabla\cdot\int \chi v f\, d^3\vec{v} - \cancel{\int f \nabla\cdot v\, \chi\, d^3\vec{v}} \to 0$$

$$- \cancel{\int f v \nabla \chi\, d^3\vec{v}} = 0$$

$$\chi = \chi(\vec{v}) \Rightarrow \nabla\chi = 0.$$

$$\vec{v} = \frac{\vec{p}}{m} \Rightarrow \nabla\cdot\vec{v} = 0.$$

$$\int \chi \vec{v}\cdot\nabla f\, d^3\vec{v} = \nabla\cdot\int \chi v f\, d^3\vec{v}$$

$\boxed{3}$

$$\int \chi \vec{F}\cdot\nabla_p f \, d^3\vec{v} =$$

$$= \underbrace{\int \nabla_p\cdot\left(F\chi_f\right) d^3\vec{v}}_{①} - \underbrace{\int \chi_f\left(\nabla_p\cdot\vec{F}\right) d^3\vec{v}}_{②} - \underbrace{\int f\left(\vec{F}\cdot\nabla_p\right)\chi \, d^3\vec{v}}_{③}$$

① $\int \nabla_p\cdot\left(\vec{F}\chi_f\right) d^3 v = F\chi_f \Big|_{V\to\infty}$

exact differential

INTEGRATING VOLUME IS ARBITRARY, SO $V\to\infty$

WITH $V\to\infty$ $f\to 0$ (no infinitely fast particles),

SO $\int \equiv 0$

② Need an assumption on force.

Say: $F = m\vec{g}$, $\vec{g} = \vec{g}(\vec{x})$

$\nabla_p\cdot\vec{g} = 0$

$\int = 0$

$$\int \chi \vec{F}\cdot\nabla_p f \, d^3 v = - \int f\left(\vec{F}\cdot\nabla_p\right)\chi \, d^3\vec{v}$$

4 Collision term:

$$\int \chi \left(\frac{\partial f}{\partial t}\right)_{coll} d\vec{v} = \frac{\partial}{\partial t}\int \chi f|_{coll} d\vec{v} - \cancel{\int f \frac{\partial \chi}{\partial t} d\vec{v}} \to 0$$

$\frac{\partial \chi}{\partial t} = 0$ as $\chi = \chi(\vec{v})$

$$\int \chi \left(\frac{\partial f}{\partial t}\right)_{coll} d\vec{v} = \frac{\partial}{\partial t}\int \chi f a|_{coll} d\vec{v}$$

All together:

$$\frac{\partial}{\partial t}\int \chi f d^3\vec{v} + \nabla\cdot\int \chi \vec{v} f d^3\vec{v} - \int f(F\cdot\nabla_p)\chi\, d^3\vec{v} = \frac{\partial}{\partial t}\int \chi f|_{coll} d\vec{v} \quad (A)$$

Define average value of χ, $\langle\chi\rangle$ as

$$\langle\chi\rangle = \frac{1}{n}\int \chi f d^3 v, \text{ or}$$

$$n\langle\chi\rangle = \int \chi f d^3 v$$

$$n\langle\chi v\rangle = \int \chi \vec{v} f d^3 v$$

$$n\langle(F\cdot\nabla_p)\chi\rangle = \int f(F\cdot\nabla_p)\chi d^3 v$$

Generalised Transport Equation:

$$\boxed{\frac{\partial}{\partial t}(n\langle\chi\rangle) + \nabla\cdot(n\langle\chi v\rangle) - n\langle(F\cdot\nabla_p)\chi\rangle = \frac{\partial}{\partial t}(n\langle\chi\rangle|_{coll})}$$

We will need "Reynolds Decomposition"

Bulk Velocity $\vec{u} = \langle \vec{V} \rangle$, "macroscopic"

$$\vec{V} = \vec{u} + \vec{c}, \text{ where } \vec{c} - \text{fluctuation "microscopic"}$$

$$\langle \vec{c} \rangle = 0.$$

$$\langle \vec{V} \rangle = \langle \vec{u} + \vec{c} \rangle$$

$\chi = m.$ $\quad \nabla_p \chi = 0$

$$\frac{\partial}{\partial t}(nm) + \nabla \cdot \left(n \langle m\vec{V} \rangle\right) = \frac{\partial}{\partial t}(nm)\Big|_{col}.$$

$\rho = n \cdot m$

$$\underline{\frac{\partial}{\partial t}\rho + \nabla(\rho \vec{u}) = S}$$

continuity equation

$\chi = m\vec{v} \quad (\chi = f(\vec{v}))$

$$\frac{\partial}{\partial t}\left(\rho\langle\vec{v}\rangle\right) + \nabla\cdot\left(\rho\langle\vec{v}\vec{v}\rangle\right) - \rho\left\langle\left(\vec{F}\cdot\nabla_p\right)\vec{v}\right\rangle = \frac{\partial}{\partial t}\left(\rho\langle\vec{v}\rangle\right)\Big|_{coll}$$

$$\frac{\partial}{\partial t}\left(\rho\langle\vec{v}\rangle\right) = \frac{\partial}{\partial t}(\rho\vec{u})$$

$$\nabla\cdot\left(\rho\langle\vec{v}\vec{v}\rangle\right) = \nabla\cdot\left(\rho\left(\vec{u}\vec{u} + \cancelto{0}{\vec{u}\langle\vec{c}\rangle} + \cancelto{0}{\langle\vec{c}\rangle\vec{u}} + \langle cc\rangle\right)\right) =$$

$$= \nabla\cdot(\rho\vec{u}\vec{u}) + \nabla\cdot(\rho\langle\vec{c}\vec{c}\rangle)$$

$-\rho\left\langle\left(\vec{F}\cdot\nabla_p\right)\vec{v}\right\rangle =$ (assume cartesian)

$$= -\rho\left\langle\left(F_x i + F_y j + F_z k\right)\left(\frac{\partial}{\partial v_x}i + \frac{\partial}{\partial v_y}j + \frac{\partial}{\partial v_z}k\right)v\right\rangle / m =$$

$$= -\frac{\rho}{m}\left\langle\left(F_x\frac{\partial}{\partial v_x} + F_y\frac{\partial}{\partial v_y} + F_z\frac{\partial}{\partial v_z}\right)\vec{v}\right\rangle =$$

$$= -\frac{\rho}{m}\left\langle F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right\rangle = -\frac{\rho}{m}\langle\vec{F}\rangle = \rho\vec{g} \quad (F = m\vec{g})$$

Coll. term:

$$\frac{\partial}{\partial t}\left(\rho\langle\vec{v}\rangle\right)\Big|_{coll} = \frac{\partial}{\partial t}\rho u\Big|_{coll} = \vec{A}$$

(Zero actually. momentum is conserved in closed system.)

So:

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\vec{u}) + \nabla\cdot\underbrace{\left(\rho\langle cc\rangle\right)}_{\text{Stress tensor } \sigma} - \frac{\rho}{m}\langle F\rangle = \vec{A}$$

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla\cdot(\rho uu) + \nabla\cdot\sigma - \frac{\rho}{m}\langle F\rangle = \vec{A}$$

momentum equation

A few words on the stress tensor:

1. coming from microscopic scales, affects macroscopic

2. Formally

$$\sigma = \underbrace{\begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}}_{\substack{\text{mechanical} \\ \text{pressure} \\ \text{(scalar)}}} + \underbrace{\begin{pmatrix} \sigma_{xx}-\pi & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\pi & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\pi \end{pmatrix}}_{\text{Stress tensor}} = -\pi \mathbb{I} + \tau$$

$$\pi = \frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$$

3. Stress tensor is a very complex thing:

Different "types" (or simplifications) of fluids exist

Newtonian

observation:

(viscous stress is linearly proportional to "shear")

$$\tau = \mu \frac{du}{dy} \quad \text{- by Newton}$$

$$\tau_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$\sigma_{ij} = p\delta_{ij} - \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

non-Newtonian

When this Does not work

Even more complex behaviour

Different Microscopic physics

GTE:

$$\frac{\partial}{\partial t}\left(n\langle\chi\rangle\right)+\nabla\cdot\left(n\langle\chi\vec{v}\rangle\right)-n\langle(F\cdot\nabla_p)\chi\rangle=\frac{\partial}{\partial t}\left(n\langle\chi\rangle\right)_{coll.}$$

$$\chi=\frac{1}{2}m\left(\vec{v}\cdot\vec{v}\right)=\frac{1}{2}mv^2$$

$$\nabla_p\chi=\frac{1}{m}\nabla_v\chi=\frac{1}{m}\left(m\nabla_v\left(\vec{v}\cdot\vec{v}\right)\frac{1}{2}\right)=$$

$$=\frac{1}{\cancel{2}}\cancel{2}\left(\vec{v}\cdot\nabla_v\right)\vec{v}=\vec{v}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\langle v^2\rangle\right)+\nabla\cdot\left(\frac{1}{2}\rho\langle v^2\vec{v}\rangle\right)-n_a\left\langle\left(F\cdot\nabla_p\right)\left(\frac{1}{2}m_a v^2\right)\right\rangle=$$

$$=\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\langle v^2\rangle\right)\Big|_{col.}=M$$

$=0$ (energy conservation)

$$\nabla\cdot\left(\frac{1}{2}\rho\langle(v\cdot v)v\rangle\right)=\qquad(v=u+c)$$

$$=\nabla\cdot\left(\frac{1}{2}\rho\langle(\vec{u}^2+2\vec{u}\vec{c}+\vec{c}^2)(\vec{u}+\vec{c})\rangle\right)=$$

$$=\nabla\cdot\left(\frac{1}{2}\rho\left(\langle u^2\vec{u}\rangle+\cancel{\langle u^2\vec{c}\rangle}^{0}+\cancel{\langle 2\vec{u}\vec{c}\vec{u}\rangle}^{0}+\langle 2\vec{u}\vec{c}\vec{c}\rangle+\langle c^2\vec{u}\rangle+\langle c^2\vec{c}\rangle\right)\right)=$$

$$=\nabla\cdot\Big(\underbrace{\frac{1}{2}\rho u^2\vec{u}}_{\text{kinetic energy}}+\underbrace{\frac{1}{2}\rho\langle c^2\rangle\vec{u}}_{\text{mechanical pressure }\pi\text{ (diagonal of }\sigma)}+\underbrace{\rho\langle\vec{c}\vec{c}\rangle}_{\sigma}\cdot\vec{u}+\underbrace{\frac{\rho}{2}\langle c^2\vec{c}\rangle}_{\text{Heat flux}}\Big)$$

Heat flux $\vec{q}=\frac{1}{2}\rho\langle c^2\vec{c}\rangle$

(Thermodynamical $\gamma=\frac{d+2}{d}$, where $d=3$ - dimensionalits of space.)

$$\frac{1}{\gamma-1}\pi+\frac{1}{2}\rho u^2=\epsilon\quad\text{- Energy density.}$$

$$n\langle\chi\rangle=\frac{1}{2}\rho\langle c^2\rangle+\frac{1}{2}\rho u^2=\frac{1}{\gamma-1}\pi+\frac{1}{2}\rho u^2,\quad\boxed{\frac{\partial n\langle\chi\rangle}{\partial t}=\frac{\partial\epsilon}{\partial t}}$$

$$\frac{\partial \epsilon}{\partial t} + \nabla\cdot(\epsilon \vec{u}) + \nabla\cdot(\sigma\cdot\vec{u}) + \vec{q} - n\langle(\vec{F}\cdot\nabla_p)\chi\rangle = \Big|_{coll.}$$

$$\| \quad n\langle F\cdot v\rangle$$

$$n\langle \vec{F}\cdot\vec{v}\rangle = n\langle F\cdot(\vec{u}+\vec{c})\rangle = n\left[\langle F\rangle\cdot u + \langle F\cdot c\rangle\right]$$

if $\vec{F}$ is velocity-independent : $\langle \vec{F}\cdot\vec{c}\rangle = \vec{F}\cdot\langle c\rangle = 0.$

$$\vec{F} = m\vec{g}$$

$$n\langle \vec{F}\cdot\vec{v}\rangle = nm\vec{g}\cdot\vec{u} = \rho\vec{g}\cdot\vec{u}$$

Energy equation:

$$\frac{\partial \epsilon}{\partial t} + \nabla\cdot(\epsilon\vec{u} + \sigma\cdot\vec{u}) + \vec{q} - \rho\vec{g}\cdot\vec{u} = 0\ (\text{coll}).$$

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finally, equations:

(continuity) $\quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0.$

(momentum) $\quad \frac{\partial(\rho\vec{u})}{\partial t} + \nabla\cdot(\rho\vec{u}\vec{u}) + \nabla\cdot\sigma - \rho\vec{g} = 0$

(Energy) $\quad \frac{\partial \epsilon}{\partial t} + \nabla\cdot(\epsilon\vec{u} + \sigma\cdot\vec{u}) + \vec{q} - \rho\vec{g}\cdot\vec{u} = 0$

(Equation of state) $\quad \epsilon \stackrel{!}{=} \frac{\Pi}{\gamma - 1} + \frac{\rho u^2}{2}$ (connects int. ϵ and mechanical pressure)

(Stress tensor generally) $\quad \sigma = \pi I + \tau$