$$\rho = \sum n_a m_a$$
$$\rho \vec{u} = \sum n_a m_a \vec{u}_a$$
$$\rho_q = \sum n_a q_a$$
$$\vec{J} = \sum n_a q_a \vec{u}_a$$

$$\sum \frac{\partial \rho_a}{\partial t} + \sum \nabla \cdot (\rho_a \vec{u}_a) = \sum \frac{\partial \rho}{\partial t}\Big|_{coll}$$

$$\downarrow$$

① $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$ (mass is conserved)

$$\sum_a \left[\frac{\partial}{\partial t} (\rho_a \vec{u}_a) + \nabla \cdot (\rho \vec{u}_a \vec{u}_a) + \nabla \cdot P_a - \frac{q_a}{m_a} \rho_a (\vec{E} + \vec{v}_a \times \vec{B}_a) = \vec{A}_a + \rho_a \vec{g} \right]$$

$$\downarrow$$

② $$\frac{\partial}{\partial t} (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \vec{u}) + \nabla \cdot P - \rho_q \vec{E} - \vec{J} \times \vec{B} - \rho \vec{g} = 0$$

(momentum is conserved)

$$\frac{\partial \epsilon_a}{\partial t} + \nabla \cdot (\epsilon_a \vec{u}_a) + \nabla \cdot (P_a \cdot \vec{u}_a) + \nabla \cdot \vec{q}_a - q_a n_a \vec{u}_a \cdot \vec{E} - \rho_a \vec{u}_a \cdot \vec{g} = 0$$

$$\epsilon_a = \frac{p_a}{\gamma - 1} + \frac{1}{2} \rho_a u_a^2 ; \quad \epsilon = \sum \epsilon_a = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2$$

energy is conserved.

③ $$\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\epsilon \vec{u}) + \nabla \cdot (P \cdot \vec{u}) + \nabla \cdot \vec{q} - \vec{J} \cdot \vec{E} - \rho \vec{u} \cdot \vec{g} = 0$$

Maxwells equations:

$$\nabla \cdot \vec{E} = \frac{\rho_a}{\varepsilon_0} \qquad (1)$$

$$\nabla \cdot \vec{B} = 0 \qquad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \qquad (3)$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \qquad (4)$$

Take $\nabla \cdot$ of (4):

$$\underbrace{\nabla \cdot \nabla \times \vec{B}}_{\text{is } 0} = \mu_0 \nabla \cdot \vec{J} + \mu_0 \varepsilon_0 \frac{\partial\, \nabla \cdot \vec{E}}{\partial t}$$

$$\boxed{\frac{\partial \rho_a}{\partial t} + \nabla \cdot \vec{J} = 0}$$

equation of current conservation.

$$F = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\frac{\partial}{\partial t}(\rho_a \vec{u}_a) + \nabla\cdot(\rho \vec{u}_a \vec{u}_a) + \nabla\cdot P_a - \frac{1}{m_a}\rho_a \langle \vec{F} \rangle = \vec{A}_a$$

Multiply by $\frac{q_a}{m_a}$, sum over a

$$\frac{\partial}{\partial t}\sum n_a q_a \vec{u}_a + \nabla\cdot\left(\sum n_a q_a \vec{u}_a \vec{u}_a\right) + \nabla\cdot\sum \frac{q_a}{m_a} P_a - \sum n_a \frac{q_a}{m_a}\langle \vec{F} \rangle =$$

$$= \sum \frac{q_a}{m_a} \vec{A}_a$$

$$\vec{u}_a = \vec{u} + \vec{w}_a$$

$$J = \sum n_a q_a \vec{u}_a = \sum n_a q_a \vec{u} + \sum n_a q_a \vec{w}_a =$$

$$= \rho_q \vec{u} + \sum n_a q_a \vec{w}_a$$

Convection current density

Conduction current density (frame moving with plasma)

$$\sum n_a q_a \vec{u}_a \vec{u}_a = \sum n_a q_a (\vec{u} + \vec{w}_a)(\vec{u} + \vec{w}_a) =$$

$$= \sum n_a q_a \vec{u}\vec{u} + \sum n_a q_a \vec{u}\vec{w}_a + \sum n_a q_a \vec{w}_a \vec{u} + \sum n_a q_a \vec{w}_a \vec{w}_a =$$

$$= \sum \underbrace{n_a q_a \vec{u}_a}_{J} \vec{u} + \sum n_a q_a \vec{u}\vec{w}_a + \sum n_a q_a \vec{w}_a \vec{w}_a =$$

$$= \vec{J}\vec{u} + \vec{u}(\vec{J} - \rho_q \vec{u}) + \sum n_a q_a \vec{w}_a \vec{w}_a =$$

$$= \vec{J}\vec{u} + \vec{u}\vec{J} - \rho_q \vec{u}\vec{u} + P_{electric}$$

$$P_{q\,total} = P + P_{q\,electric}$$

So,

$$\frac{\partial \vec{J}}{\partial t} + \nabla\cdot\left(\vec{J}\vec{u} + \vec{u}\vec{J} - \rho_q \vec{u}\vec{u} + P_q\right) - n_a \frac{q_a}{m_a}\langle \vec{F} \rangle = \sum \frac{q_a}{m_a}\vec{A}_a$$

Simplifying:

$$\cancel{\nabla\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)}$$

$$\mu_0 \nabla\cdot\vec{J} + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\left(\nabla\cdot\vec{E}\right) = 0.$$

$$\frac{J}{\cancel{L}} = \frac{\varepsilon_0 E}{\cancel{L}T} \qquad \vec{J} = \sigma\vec{E} \text{ - Ohm's Law}$$

$$\sigma \cancel{E} = \frac{\varepsilon_0 \cancel{E}}{T} \qquad T = \frac{\varepsilon_0}{\sigma} \approx 10^{-11}$$

$T \gg 10^{-11}$ — $\frac{\partial E}{\partial t}$ can be neglected.

$$\nabla\times\vec{B} = \mu_0\left(\vec{J} \cancel{+ \epsilon_0 \frac{\partial E}{\partial t}}\right) \longrightarrow \nabla\times\vec{B} = \mu_0\vec{J}$$

$$\mu_0\vec{J}\times\vec{B} = \left(\nabla\times\vec{B}\right)\times\vec{B} =$$

$$= \left(\vec{B}\cdot\nabla\right)\vec{B} - \underbrace{\nabla\frac{B^2}{2}}_{\text{magnetic pressure}}$$

$$\nabla\cdot\left(\vec{B}\vec{B}\right) = \left(\vec{B}\cdot\nabla\right)\vec{B} + \vec{B}\underbrace{\left(\nabla\cdot\vec{B}\right)}_{=0} = \left(\vec{B}\cdot\nabla\right)\vec{B}$$

$$\nabla\frac{B^2}{2} = \nabla\left(\frac{B^2}{2}\cdot \underbrace{\mathbb{I}}_{\text{unit}}\right)$$

Now, momentum eq:

$$\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot \left(\rho \vec{u}\vec{u} + P + \frac{B^2}{2\mu_0} - \frac{1}{\mu_0}\vec{B}\vec{B} \right) - \rho_q \vec{E} - \rho \vec{g} = 0 ,$$

Assume isotropic pressure

$$\nabla \cdot P = \nabla \cdot (pI) = \nabla p$$

Charge neutrality —

— see home work.

$\rho_q = 0$!

$m_i \gg m_e$ ions are heavier then electrons

Charge pressure can be neglected if electron pressure is small

$P_q = 0$

Induction equation can be derived from collision term in ⊗ and ME.

We use simplified derivation.

$$\vec{J} = \sigma\left(\vec{E} + \vec{v}\times\vec{B}\right)$$ — Ohm's law from ⊗

$$\nabla\times B = \mu_0 J$$

$$\nabla\times\vec{B} = \mu_0\sigma\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$\nabla\times\left(\nabla\times\vec{B}\right) = \mu_0\sigma\left(\nabla\times\vec{E} + \nabla\times\left(\vec{v}\times\vec{B}\right)\right)$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\left(\nabla\times\vec{B}\right) = \mu_0\sigma\left(-\frac{\partial\vec{B}}{\partial t} + \nabla\times\left(\vec{v}\times\vec{B}\right)\right)$$

$$\frac{\partial B}{\partial t} = \nabla\times\left(\vec{v}\times\vec{B}\right) - \frac{1}{\mu_0\sigma}\underbrace{\nabla\times\left(\nabla\times\vec{B}\right)}_{=\nabla(\underbrace{\nabla\cdot\vec{B}}_{\simeq 0}) - \nabla^2 B}$$

$$\frac{\partial B}{\partial t} = \nabla\times\left(\vec{v}\times\vec{B}\right) + \frac{1}{\mu_0\sigma}\nabla^2 B$$

σ is large (conducting), can be neglected.

however responsible for reconnection → solar flares etc

Summary:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

$$\frac{\partial \rho\vec{u}}{\partial t} + \nabla\cdot\left(\rho\vec{u}\vec{u} + \left(p + \frac{B^2}{2\mu_0}\right)\vec{I} - \frac{1}{\mu_0}\vec{B}\vec{B}\right) - \rho\vec{g} = 0.$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot\left(\left(\mathcal{E} + p + \frac{B^2}{2\mu_0}\right)\vec{u} + \vec{q} - \frac{1}{\mu_0}(\vec{u}\cdot\vec{B})\vec{B}\right) - \rho\vec{u}\cdot\vec{g} = 0.$$

$$\frac{\partial B}{\partial t} = \nabla\times(\vec{v}\times\vec{B})$$

$$\mathcal{E} = \frac{\rho u^2}{2} + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}$$

$$\nabla\cdot(\vec{B}\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{B} + \vec{B}(\nabla\cdot\vec{B})$$

Static: $\nabla\left(p + \frac{B^2}{2\mu_0}\right) - (\vec{B}\cdot\nabla)\vec{B} = \rho\vec{g}$