Generalized transport equation for $\chi = \chi(\vec{v})$

$$\frac{\partial}{\partial t}\left(n_a\langle\chi\rangle_a\right) + \nabla\cdot\left(n_a\langle\chi\vec{v}\rangle_a\right) - n_a\langle\left(\vec{F}\cdot\nabla_p\right)\chi\rangle = $$
$$= \frac{\partial}{\partial t}\left(n_a\langle\chi\rangle_a\right)\Big|_{col}$$

(1) $\chi = m_a$

Reynolds decomposition:

Define bulk velocity $\vec{u}_a = \langle\vec{v}_a\rangle$ "macroscopic"

Then $\vec{v}_a = \vec{u}_a + \vec{c}_a$, where $\vec{c}_a$ is fluctuation of velocity.

Note $\langle c_a\rangle = 0$; $\langle\vec{v}_a\rangle = \langle\vec{u}_a + \vec{c}_a\rangle$

$\chi = m_a \Rightarrow \nabla_v\chi = 0$.

GTE:
$$\frac{\partial}{\partial t}\left(n_a m_a\right) + \nabla\cdot\left(n_a\langle m_a\vec{v}\rangle_a\right) = \frac{\partial}{\partial t}\left(n_a m_a\right)_{col}$$
$$\frac{\partial}{\partial t}\left(n_a m_a\right) + \nabla\cdot\left(n_a m_a\vec{u}_a\right) = \frac{\partial}{\partial t}\left(n_a m_a\right)_{col}$$

$\rho_a = n_a m_a$ – density of a-th component.

$$\boxed{\frac{\partial\rho_a}{\partial t} + \nabla\cdot\left(\rho\vec{v}_a\right) = \left(\frac{\partial\rho_a}{\partial t}\right)_{coll.}} = S_a$$

~~the~~ Continuity equation

(2) $\chi = m_a \vec{v}$

$$\frac{\partial}{\partial t}\left(\rho_a \langle \vec{v} \rangle_a\right) + \nabla \cdot \left(\rho_a \langle \vec{v}\vec{v} \rangle_a\right) - \rho_a \langle (\vec{F} \cdot \nabla_p) \vec{v} \rangle_a =$$

$$= \frac{\partial}{\partial t}\left(\rho_a \langle \vec{v} \rangle_a\right)\Big|_{col}$$

$\vec{v}_a = \vec{u}_a + \vec{c}_a \qquad \langle \vec{c}_a \rangle = 0$

$$\frac{\partial}{\partial t}\left(\rho_a \langle \vec{v} \rangle_a\right) = \frac{\partial}{\partial t}\left(\rho_a \vec{u}_a\right)$$

$$\nabla \cdot \left(\rho_a \langle \vec{v}\vec{v} \rangle_a\right) = \nabla \cdot \left(\rho_a \left(\vec{u}_a \vec{u}_a + \underbrace{\vec{u}_a \overset{=0}{\langle \vec{c}_a \rangle} + \overset{=0}{\langle \vec{c}_a \rangle} \vec{u}_a}_{=0} + \langle \vec{c}_a \vec{c}_a \rangle\right)\right) =$$

$$= \nabla \cdot \left(\rho_a \left(\vec{u}_a \vec{u}_a + \langle \vec{c}_a \vec{c}_a \rangle\right)\right)$$

$$-\rho_a \langle (\vec{F} \cdot \nabla_p) \vec{v} \rangle_a = -\rho_a \left\langle \left(F_x \vec{i} + F_y \vec{j} + F_z \vec{k}\right) \cdot \left(\frac{\partial}{\partial v_x}\vec{i} + \frac{\partial}{\partial v_y}\vec{j} + \frac{\partial}{\partial v_z}\vec{k}\right) \vec{v} \right\rangle / m_a =$$

$\boxed{\nabla_p = \nabla_v / m_a}$

$$= -\frac{\rho_a}{m_a} \left\langle \left(F_x \frac{\partial}{\partial v_x} + F_y \frac{\partial}{\partial v_y} + F_z \frac{\partial}{\partial v_z}\right) \vec{v} \right\rangle =$$

$$= -\frac{\rho_a}{m_a} \langle F_x \vec{i} + F_y \vec{j} + F_z \vec{k} \rangle = -\frac{\rho_a}{m_a} \langle \vec{F} \rangle$$

Collision term:

$$\frac{\partial}{\partial t}\left(\rho_a \langle \vec{v} \rangle_a\right)\Big|_{coll} = \left(\frac{\partial}{\partial t} \rho_a \vec{u}_a\right)_{coll} = \vec{A}_a$$

$$\frac{\partial}{\partial t}\left(\rho_a \vec{u}_a\right) + \nabla \cdot \left(\rho \vec{u}_a \vec{u}_a\right) + \nabla \cdot \left(\rho \langle \vec{c}_a \vec{c}_a \rangle\right) - \frac{\rho_a}{m_a} \langle \vec{F} \rangle = \vec{A}_a$$

Pressure tensor $\quad P_a = \rho \langle \vec{c}_a \vec{c}_a \rangle$ (viscosity is here!)

$$\frac{\partial}{\partial t}\left(\rho_a \vec{u}_a\right) + \nabla \cdot \left(\rho \vec{u}_a \vec{u}_a\right) + \nabla \cdot P_a - \frac{\rho_a}{m_a} \langle \vec{F} \rangle = \vec{A}_a$$

if $\vec{F} = q_a (\vec{E} + \vec{u}_a \times \vec{B}) + \rho \vec{g}$

$$\boxed{\frac{\partial}{\partial t}\left(\rho_a \vec{u}_a\right) + \nabla \cdot \left(\rho \vec{u}_a \vec{u}_a\right) + \nabla \cdot P_a - \frac{q_a}{m_a} \rho_a \left(\vec{E} + \vec{u}_a \times \vec{B}_a\right) = \vec{A}_a + \rho \vec{g}}$$

Momentum equation

$$② \quad \frac{\partial}{\partial t}(\rho_a \vec{u}_a) + \nabla\cdot(\rho \vec{u}_a \vec{u}_a) + \nabla\cdot P_a - \frac{q_a}{m_a}\rho_a(\vec{E} + \vec{u}_a \times \vec{B}_a) = \vec{A}_a + \rho_a \vec{g}$$

$$\nabla\cdot(\rho \vec{u}_a \vec{u}_a) = (\rho \vec{u}_a \cdot \nabla)\vec{u}_a + \vec{u}_a(\nabla \cdot \rho \vec{u}_a) =$$

$$= \rho_a(\vec{u}_a \cdot \nabla)\vec{u}_a + \vec{u}_a(\nabla \cdot \rho \vec{u}_a)$$

$$\vec{u}_a \frac{\partial \rho_a}{\partial t} + \rho_a \frac{\partial \vec{u}_a}{\partial t} + \rho_a(\vec{u}_a \cdot \nabla)\vec{u}_a + \underbrace{\vec{u}_a(\nabla\cdot\rho\vec{u}_a)} + \ldots$$

$$\rightarrow u_a\left(\frac{\partial \rho_a}{\partial t} + \nabla\cdot\rho\vec{u}_a\right) = \vec{u}_a S_a$$

So, ② using ① (continuity equation) can be rewritten as

$$\boxed{\rho_a \underbrace{\left(\frac{\partial \vec{u}_a}{\partial t} + (\vec{u}_a\cdot\nabla)\vec{u}_a\right)}_{\frac{D\vec{u}_a}{Dt}} + S_a \vec{u}_a + \nabla\cdot P_a - \frac{q_a}{m_a}\rho_a(\vec{E} + \vec{u}_a\times\vec{B}_a) = \vec{A}_a + \rho_a\vec{g}}$$

Momentum eq. 2

(3)

$$\frac{\partial}{\partial t}\left(n_a\langle\chi\rangle_a\right)+\nabla\cdot\left(n_a\langle\chi\vec{v}\rangle_a\right)-n_a\langle\vec{F}\cdot\nabla_p\rangle\chi\rangle=\frac{\partial}{\partial t}\left(n_a\langle\chi\rangle_a\right)_{col}$$

$$\boxed{\chi=\frac{1}{2}m_a v^2}=\frac{1}{2}m_a\left(\vec{v}\cdot\vec{v}\right)$$

$$\nabla_p\chi=\frac{1}{m_a}\nabla_v\chi=\frac{1}{m_a}\left(m_a(\vec{v}\cdot\vec{v})\cdot\frac{1}{2}\right)=\frac{1}{2}2\left(\vec{v}\cdot\nabla_v\right)\vec{v}=\vec{v}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\langle v^2\rangle_a\right)+\nabla\cdot\left(\frac{1}{2}\rho_a\langle v^2\vec{v}\rangle_a\right)-\overbrace{n_a\langle(F\cdot\nabla_p)\left(\frac{1}{2}m_a v^2\right)\rangle}^{n_a(\vec{F}\cdot\vec{v})}=$$

$$=\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_a\langle v^2\rangle_a\right)\Big|_{col}}_{M_a}$$

$$\nabla\cdot\left(\frac{1}{2}\rho_a\langle(\vec{v}\cdot\vec{v})\vec{v}\rangle_a\right)=$$

$$\nabla\cdot\left(\frac{1}{2}\rho_a\langle\left((\vec{u}_a+\vec{c}_a)\cdot(\vec{u}_a+\vec{c}_a)\right)(\vec{u}_a+\vec{c}_a)\rangle_a\right)=$$

$$\nabla\cdot\left(\frac{1}{2}\rho_a\langle\left(u_a^2+2\vec{u}_a\cdot\vec{c}_a+c_a^2\right)(\vec{u}_a+\vec{c}_a)\rangle_a\right)=$$

$$=\nabla\cdot\left(\frac{1}{2}\rho_a\left(\langle u_a^2\vec{u}_a\rangle+\langle u_a^2\underbrace{\vec{c}_a}_{0}\rangle+\langle 2\vec{u}_a\underbrace{\vec{c}_a}_{0}\vec{u}_a\rangle+\langle 2\vec{u}_a\vec{c}_a\vec{c}_a\rangle\right.\right.$$

$$\left.\left.+\langle c_a^2 u_a\rangle+\langle c_a^2\vec{c}_a\rangle\right)\right)=$$

$$=\nabla\cdot\left(\frac{1}{2}\rho_a u_a^2\vec{u}_a+\frac{1}{2}\rho_a\langle c_a^2\rangle\vec{u}_a+\rho_a\langle\vec{c}_a\vec{c}_a\rangle\cdot\vec{u}_a+\frac{\rho_a}{2}\langle c_a^2\vec{c}_a\rangle\right)$$

Heat flux $\vec{q}_a=\frac{1}{2}\rho_a\langle c_a^2\vec{c}_a\rangle$

Pressure tensor:

$$P_a = \rho_a \langle \vec{c}_a \vec{c}_a \rangle$$

Scalar pressure is the trace of P_a

$$p_a = \frac{1}{d} \sum_{ij} P_{ij} \delta_{ij} = \frac{1}{d} \sum_{i} P_{ii}$$

$d = 3$ – dimensionality of space

$$p_a = \frac{1}{d} \rho_a \sum \langle c_{ai}^{2} \rangle = \frac{1}{d} \rho_a \langle c_a^{2} \rangle$$

Or with γ (adiabatic index is connected to the number of degrees of freedom. For monatomic gas $d = d_f = 3$ $\gamma = \frac{\gamma + 2}{\gamma}$ $\cancel{4}$).

$$p_a = \frac{1}{d} \rho_a \langle c_a^2 \rangle = \frac{\gamma - 1}{2} \rho_a \langle c_a^2 \rangle$$

$$\chi = \frac{1}{2} m_a (\vec{v}\cdot\vec{v}) \quad , \quad \vec{v} = \vec{u} + \vec{c}$$

$$n_a \langle \chi \rangle_a = \underbrace{\frac{1}{2}\rho_a \langle c_a^2 \rangle}_{p_a/(\gamma-1)} + \frac{1}{2}\rho_a u_a^2 =$$

$$= \frac{p_a}{\gamma - 1} + \frac{1}{2}\rho_a u_a^2 = \epsilon_a \text{ - energy density}$$

$$\nabla \cdot \Big(\frac{1}{2}\rho_a u_a^2 \vec{u}_a + \frac{1}{2}\rho_a \langle c_a^2 \rangle \vec{u}_a + \underbrace{\rho_a \langle \vec{c}_a \vec{c}_a \rangle}_{P} \cdot \vec{u}_a + \frac{\rho_a}{2} \langle c_a^2 \vec{c}_a \rangle \Big) =$$

$$= \nabla \cdot \Big(\epsilon_a \vec{u}_a + P_a \cdot \vec{u}_a + \vec{q}_a \Big)$$

Energy eq:

$$\frac{\partial \epsilon_a}{\partial t} + \nabla \cdot (\epsilon_a \vec{u}_a) + \nabla \cdot (P_a \cdot u_a) + \nabla \cdot \vec{q}_a - n_a \langle \vec{F} \cdot \vec{v} \rangle_a = M_a$$

$$\langle \vec{F} \cdot \vec{v} \rangle = \langle \vec{F} \cdot (\vec{u}_a + \vec{c}_a) \rangle = \langle \vec{F} \rangle \cdot \vec{u}_a + \langle \vec{F} \cdot \vec{c}_a \rangle$$

for $\vec{F} \neq \vec{F}(\vec{v})$ $\quad \langle \vec{F} \cdot \vec{c}_a \rangle = \vec{F} \cdot \langle \vec{c}_a \rangle = 0.$

for Lorentz force

$$\langle \vec{F} \cdot \vec{c}_a \rangle = \Big\langle \Big(q_a \big(\underbrace{\vec{E}}_{\text{zero}} + \vec{v} \times \vec{B} \big) + \underbrace{m\vec{g}}_{\text{zero}} \Big) \cdot \vec{c}_a \Big\rangle =$$

$$= q_a \langle ((\vec{u}_a + \vec{c}_a) \times \vec{B}) \cdot \vec{c}_a \rangle =$$

$$= q_a (\vec{u}_a \times \vec{B}) \overset{0}{\langle \vec{c}_a \rangle} + q_a \underbrace{\langle (\vec{c}_a \times \vec{B}) \cdot \vec{c}_a \rangle}_{=0} = 0.$$

EE:

$$\boxed{\frac{\partial \epsilon_a}{\partial t} + \nabla \cdot (\epsilon_a \vec{u}_a) + \nabla \cdot (P_a \cdot u_a) - q_a n_a \vec{u}_a \cdot \vec{E} - \rho_a \vec{u}_a \cdot \vec{g} = M_a}$$