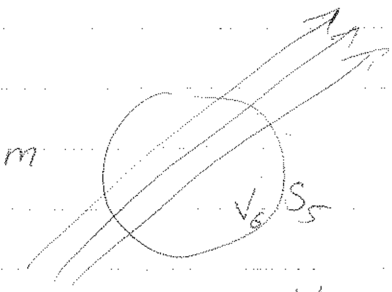
$$f(\vec{r},\vec{p},t) = f(r_i, p_i, t)$$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r_i}dr_i + \frac{\partial f}{\partial p_i}dp_i$$ — full differential

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f_i}{\partial r_i}\underbrace{\frac{dr_i}{dt}}_{v_i = \frac{p_i}{m}} + \frac{\partial f}{\partial p_i}\underbrace{\frac{dp_i}{dt}}_{F_i = a_i \cdot m}$$



$$\frac{\partial}{\partial t}\int_{V_6} f\, dV_6 + \int_{S_5} f v_i n_i \, dS_5 = 0.$$ — conservation of particle number

$$\frac{\partial}{\partial t}\int_{V_6} f\, dV_6 + \int_{V_6}\frac{\partial}{\partial x_i}(f v_i)\, dV_6 = 0$$ — Gauss theorem

(1) $$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(f \cdot v_i) = 0.$$

Splitting i into x and v components:

$$\frac{\partial}{\partial t}f(x_i,p_i,t) + \frac{\partial}{\partial x_i}\left(f(x_i,p_i,t)\cdot\frac{dx_i}{dt}\right) + \frac{\partial}{\partial p_i}\left(f(x_i,p_i,t)\frac{dp_i}{dt}\right) = 0$$

Liouville eq-n.

Liouville eq:

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial x_i}\left(f \cdot \frac{dx_i}{dt}\right) + \frac{\partial}{\partial p_i}\left(f \cdot \frac{dp_i}{dt}\right) = 0.$$

$$\frac{\partial}{\partial t} f + \frac{dx_i}{dt}\frac{\partial}{\partial x_i} f + \frac{dp_i}{dt}\frac{\partial}{\partial p_i} f + \underbrace{f\left[\frac{\partial}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial}{\partial p_i}\frac{dp_i}{dt}\right]} = 0.$$

$$\left[\frac{\partial}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial}{\partial p_i}\frac{dp_i}{dt}\right] = \frac{\partial}{\partial x_i}\dot{x}_i + \frac{\partial}{\partial p_i}\dot{p}_i = 0$$

Hamilton equations (closed system)

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \; ; \; \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

substitute

$$\frac{\partial}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial H}{\partial x_i} = 0.$$

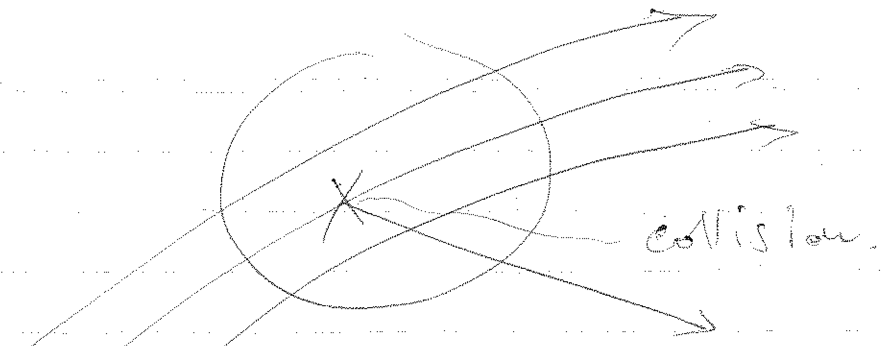
note: 6D-divergence of 6D-flow in phase space = 0!

6D-divergence of 6-velocity

Liouville eq:

$$\frac{\partial}{\partial t} f + v_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial p_i} = 0.$$

Collisions:



Formally:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial p_i} = \left(\frac{\partial f}{\partial t}\right)_{coll.}$$

Complicated thing coming from statistical mechanics

Species "a": f^a

$$\frac{\partial f^a}{\partial t} + v_i \frac{\partial f^a}{\partial x_i} + F_i \frac{\partial f^a}{\partial p_i} = \left(\frac{\partial f^a}{\partial t}\right)_{coll.}$$

In vector form: Boltzmann equation.

$$\frac{\partial f^a}{\partial t} + \vec{V} \cdot \nabla f_a + \vec{F} \cdot \nabla_p f_a = \left(\frac{\partial f^a}{\partial t}\right)_{coll.} \qquad (*)$$

What we want to do: we want to get rid of velocity part of space, so we integrate $(*)$ over an arbitrary volume in V-space $d^3\vec{V}$.

We also multiply BE by some function of $\vec{V}$, $X(\vec{V})$

$$\frac{\partial f^a}{\partial t} + \vec{v}\cdot\nabla f^a + \vec{F}\cdot\nabla_p f^a = \left(\frac{\partial f}{\partial t}\right)_{coll.} \quad ; \boxed{\chi = \chi(\vec{v})}$$

$$\chi\frac{\partial f^a}{\partial t} + \chi\vec{v}\cdot\nabla f^a + \chi\vec{F}\nabla_p f^a = \chi\left(\frac{\partial f^a}{\partial t}\right)_{coll}$$

$$\underset{(1)}{\int\chi\frac{\partial f^a}{\partial t}d^3\vec{v}} + \underset{(2)}{\int\chi\vec{v}\cdot\nabla f^a d^3\vec{v}} + \underset{(3)}{\int\chi F\nabla_p f^a d^3\vec{v}} = \underset{(4)}{\int\chi\left(\frac{\partial f}{\partial t}\right)_{coll}d^3\vec{v}}$$

$\boxed{1}$

$$\int\chi\frac{\partial f^a}{\partial t}d^3v = \frac{\partial}{\partial t}\int\chi f^a d^3v - \int\frac{\partial\chi}{\partial t}f^a d^3v$$

$$\chi \equiv \chi(\vec{v}) \Rightarrow \frac{\partial\chi}{\partial t} = 0 \Rightarrow$$

$$\Rightarrow \int\frac{\partial\chi}{\partial t}f^a d^3v = 0$$

$$\Downarrow$$

$$\boxed{\int\chi\frac{\partial f^a}{\partial t}d^3v = \frac{\partial}{\partial t}\int\chi f^a d^3v}$$

$\boxed{2}$

$$\int\chi\vec{v}\cdot\nabla f^a d^3\vec{v} =$$

$$= \nabla\cdot\int\chi v f^a d^3v - \int f^a\,\nabla\cdot v\,\chi d^3\vec{v} -$$

$$- \int f^a v\cdot\nabla\chi d^3\vec{v}$$

$$\chi \equiv \chi(\vec{v}) \Rightarrow \nabla\chi = 0 \Rightarrow \boxed{\int f^a v\cdot\nabla\chi d^3\vec{v} = 0.}$$

$\vec{v} = \frac{\vec{p}}{m_a} \Rightarrow \vec{v} \equiv f(\vec{p})$ does not depend on x.

$$\nabla\cdot\vec{v} = 0 \Rightarrow \int f^a\chi\,\nabla\cdot v\,d^3\vec{v} = 0.$$

$$\boxed{\int\chi\vec{v}\cdot\nabla f^a d^3\vec{v} = \nabla\cdot\int\chi v f^a d^3v}$$

$\nabla_p = m_a \nabla_v$

3

$$\int \chi \vec{F}\cdot\nabla_p f^a d^3\vec{v} =$$

$$\underbrace{\int \nabla_p\cdot\left(\vec{F}\chi f^a\right)}_{1} d^3\vec{v} - \int \chi f^a \underbrace{\left(\nabla_p\cdot\vec{F}\right)}_{2} d^3\vec{v} -$$

$$- \int f^a \left(\vec{F}\cdot\nabla_p\right)\chi \underbrace{d^3\vec{V}}_{3}$$

1 $\int \nabla_p\cdot\left(\vec{F}\chi f^a\right) d^3\vec{V}$ — exact differential

$$\int \nabla_p\left(\vec{F}\chi f_a\right) d^3\vec{v} = F\chi f_a \Big|_{v\to\infty}$$

Integrating volume is arbitrary, so can be chosen $v \to \infty$.
With $v\to\infty$ $f \to 0$ (there are no infinitely fast particles).
Thus $\int \nabla_p\cdot(\vec{F}\chi f^a)\, d^3\vec{v} = 0$.

$$\int \chi f^a \left(\nabla_p \cdot \vec{F}\right) d^3\vec{v} = 0$$

Say, Lorentz force + gravity

$\vec{F} = q_a\left(\vec{E} + \vec{v}\times\vec{B}\right) + m_a\vec{g}$

$\vec{E} = \vec{E}(\vec{x})$; $\vec{g} = \vec{g}(\vec{x})$;

$\nabla_p \vec{E} = 0$; $\nabla_p \vec{g} = 0$

$\vec{v}\times\vec{B}$ perpendicular to $\vec{v}$ and $\vec{B}$

$\nabla_p\cdot\left(\vec{v}\times\vec{B}\right) = 0$

$\nabla_p\cdot\vec{F} = 0$

$$\boxed{\int \chi\vec{F}\cdot\nabla_p f^a d^3\vec{v} = -\int f^a\left(\vec{F}\cdot\nabla_p\right)\chi\, d^3\vec{V}}$$

4 Collision term

$$\int \chi \left(\frac{\partial f^a}{\partial t}\right)_{col} d\vec{v} = \frac{\partial}{\partial t}\int \chi f^a|_{col} d\vec{v} - \int f^a \frac{\partial \chi}{\partial t} d\vec{v}$$

$$\chi = \chi(\vec{v}) \Rightarrow$$

$$\Rightarrow \int f^a \frac{\partial \chi}{\partial t} d\vec{v} = 0$$

$$\int \chi \left(\frac{\partial f^a}{\partial t}\right)_{col} d\vec{v} = \frac{\partial}{\partial t}\int \chi f^a|_{col} d\vec{v}$$

All together:

$$\boxed{\frac{\partial}{\partial t}\int \chi f^a d^3v + \nabla\cdot\int \chi \vec{v} f^a d^3v - \int f^a (\vec{F}\cdot\nabla_p)\chi\, d^3\vec{v} = \frac{\partial}{\partial t}\int \chi f^a|_{col} d\vec{v}} \quad (A)$$

Define average value of χ, $\langle \chi \rangle$ as

$$\boxed{\langle \chi \rangle_a = \frac{1}{n_a}\int \chi f^a d^3v} \quad \text{or} \quad \boxed{n_a \langle \chi \rangle_a = \int \chi f^a d^3v}$$

$$n_a \langle \chi \vec{v} \rangle_a = \int \chi \vec{v} f^a d^3v$$

Substitute into (A):

$$n_a \langle (\vec{F}\cdot\nabla_p)\chi \rangle = \int f^a (\vec{F}\cdot\nabla_p)\chi\, d^3\vec{v}$$

$$\boxed{\frac{\partial}{\partial t}(n_a\langle \chi \rangle_a) + \nabla\cdot(n_a \langle \chi v \rangle_a) - n_a\langle (\vec{F}\cdot\nabla_p)\chi \rangle = \frac{\partial}{\partial t}(n_a \langle \chi \rangle_a|_{col})}$$

Generalised Transport Equation