Schwartzschild criterion

$$P_1' V_1'^{\gamma} = P_2' V_2'^{\gamma}$$

$$\frac{P_1'}{P_2'} = \frac{V_2'^{\gamma}}{V_1'^{\gamma}} = \left(\frac{V_1'}{V_2'}\right)^{-\gamma}$$

$$M_1' = M_2'$$

$$\rho_1' = \frac{M_1'}{V_1'} ; \rho_2' = \frac{M_1'}{V_2'}$$

$$\frac{P_1'}{P_2'} = \left(\frac{V_2'}{V_1'}\right)^{\gamma} = \left(\frac{\rho_1'}{\rho_2'}\right)^{\gamma}$$

$$\frac{\rho_1'}{\rho_2'} = \frac{V_2'}{V_1'}$$

$$\frac{\rho_1'}{\rho_2'} = \left(\frac{P_1'}{P_2'}\right)^{1/\gamma} ; \rho_2' = \rho_1'\left(\frac{P_2'}{P_1'}\right)^{1/\gamma} < \rho_2$$

$$\rho_1'\left(\frac{P_2'}{P_1'}\right)^{1/\gamma} < \underset{\text{external}}{\rho_2}$$ — convective inst. condition.

Now, $\rho_1' = \rho_1 = \rho$, $P_1' = P_1 = P$ (internal = external at the bottom)

$P_2' = P_2$ (pressures equal at the top after it rises).

$$\rho_2' = \rho_1' + d\rho' \qquad \rho_2' = \rho + d\rho$$

$$P_2' = P_1' + dP' \qquad \rho_1' = \rho$$

$$P_2' = P + dP$$

$$P_1' = P_1 = P$$

$$\boxed{\rho\left(\frac{P+dP}{P}\right)^{1/\gamma} < (\rho + d\rho)} \qquad \left(\frac{P+dP}{P}\right)^{1/\gamma} < \frac{\rho + d\rho}{\rho}$$

$$\left(1 + \frac{dP}{P}\right)^{1/\gamma} < 1 + \frac{d\rho}{\rho} \qquad (1+\alpha)^n \sim 1 + n\alpha \text{ for small } \alpha$$

$$\cancel{1} + \frac{1}{\gamma}\frac{dP}{P} < \cancel{1} + \frac{d\rho}{\rho} ; \quad \boxed{\frac{1}{\gamma}\frac{dP}{P} < \frac{d\rho}{\rho}}$$

for radially dependent P, ρ divide by dr

$$\frac{1}{\gamma}\frac{1}{P_r}\frac{dP_r}{dr} < \frac{1}{\rho_r}\frac{d\rho_r}{dr}$$

$$P = \rho R T \; ; \quad \rho = \frac{P}{RT} \; ; \quad \frac{d\rho}{dr} = \frac{\frac{dP}{dr}\cdot RT - R\frac{dT}{dr}P}{R^2T^2} =$$

$$= \frac{\frac{dP}{dr}T - \frac{dT}{dr}P}{RT^2} \; ;$$

$$\frac{1}{\rho_r}\frac{d\rho_r}{dr} = \frac{\cancel{R}\cancel{T}}{P}\left(\frac{\frac{dP}{dr}T - \frac{dT}{dr}P}{RT^{\cancel{2}}}\right) =$$

$$= \frac{\frac{dP}{dr}T - \frac{dT}{dr}P}{PT} = \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}$$

$$\frac{1}{\gamma}\frac{1}{P_r}\frac{dP_r}{dr} < \frac{1}{P_r}\frac{dP_r}{dr} - \frac{1}{T_r}\frac{dT_r}{dr}$$

$$\frac{1}{T_r}\frac{dT_r}{dr} < \frac{1}{P_r}\frac{dP_r}{dr} - \frac{1}{\gamma}\frac{1}{P_r}\frac{dP_r}{dr}$$

$$\frac{1}{T_r}\frac{dT_r}{dr} < \left(1 - \frac{1}{\gamma}\right)\frac{1}{P_r}\frac{dP_r}{dr}$$

$$\frac{dT_r}{dr} < \left(1 - \frac{1}{\gamma}\right)\frac{T_r}{P_r}\frac{dP_r}{dr}$$

$$\frac{dT_r}{dr} < 0 \; ; \quad \frac{dP_r}{dr} < 0$$

$$\boxed{\left|\frac{dT_r}{dr}\right| > \left(1 - \frac{1}{\gamma}\right)\frac{T_r}{P_r}\left|\frac{dP_r}{dr}\right|}$$

Schwartzschild criterion.