By 28th of April:

(1) Lax-Friedrichs scheme, same as the previous problem, but 2D, Kelvin-Helmholtz instability. Use any programming language and the results of your previous homework to solve numerically the following:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)=k_1\left(\frac{\partial^2 \rho}{\partial x^2}+\frac{\partial^2 \rho}{\partial y^2}\right),$$

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial}{\partial x}(\rho v_x v_x+p)+\frac{\partial}{\partial y}(\rho v_x v_y)=k_2\frac{\partial^2}{\partial x^2}(\rho v_x),$$

$$\frac{\partial(\rho v_y)}{\partial t}+\frac{\partial}{\partial x}(\rho v_x v_y)+\frac{\partial}{\partial y}(\rho v_y v_y+p)=k_2\frac{\partial^2}{\partial y^2}(\rho v_y),$$

$$\frac{\partial \varepsilon}{\partial t}+\frac{\partial}{\partial x}(v_x(p+\varepsilon))+\frac{\partial}{\partial y}(v_y(p+\varepsilon))=k_3\left(\frac{\partial^2 \varepsilon}{\partial x^2}+\frac{\partial^2 \varepsilon}{\partial y^2}\right),$$

where the diffusion coefficients $k_1=kD_1$, $k_2=kD_2$, $k_3=kD_3$, $k=\frac{\Delta x^2}{4\Delta t}$, and D_1, D_2, D_3 are factors less than 1 introduced to reduce diffusivity of the solution.

Determine the time step Δt using Courant-Friedrichs-Lewy criterion $\Delta t = C\frac{\Delta x}{v_{\max}}$ at each time step, where C is the safety factor less than 1 (e.g. C=0.3). To calculate v_{max}, sound speed $c_s=\sqrt{\frac{\gamma p}{\rho}}$ has to be taken into account, so $v_{max}=\max(|v|, c_s)$ over the computational domain. Use the same as before equation of state to close the system:

$$\varepsilon=\frac{p}{\gamma-1}+\frac{\rho\left(v_x^2+v_y^2\right)}{2},$$

where $\gamma=4/3$. Set the physical domain size $x,y=0..1$, and periodic boundary conditions. As the initial condition, use:

$$y<0.4+0.01\cdot\sin(2\pi x):\rho=1;p=2.5;v_x=0.5$$

$$0.4+0.01\cdot\sin(2\pi x)\ge y\ge 0.6+0.01\cdot\sin(2\pi x):\rho=2;p=2.5;v_x=-0.5$$

$$y>0.6+0.01\cdot\sin(2\pi x):\rho=1;p=2.5;v_x=0.5$$

As the scheme is quite diffusive, you will need to use high resolution (I used 256x256 in the pictures below). Since the instability takes time to develop, you will need a few thousands of time steps. Try reducing the diffusivity (I found that $D_1=D_3=0.02$, and $D_2=0.5$ work very well).

Plot the evolution of the solution (density, velocity, internal energy, pressure) on time. Make a movie, if you can. If the language you use allows, and you know how to do it, try estimating computational time (in seconds) your computer needs for a single time step.