By 21th of April:

(1) Lax-Friedrichs scheme. Implement full one-dimensional hydrodynamic solver and test it with the Sod shock tube. Use any programming language and the results of your previous homework to solve numerically the following system of one-dimensional equations with additional diffusive terms:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) = k\frac{\partial^2}{\partial x^2}\rho,$$

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial}{\partial x}(\rho v_x v_x + p) = k\frac{\partial^2}{\partial x^2}(\rho v_x),$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x}(v_x(p+\varepsilon)) = k\frac{\partial^2 \varepsilon}{\partial x^2},$$

where the diffusion coefficient k is defined as $k = \frac{\Delta x^2}{2\Delta t}$. Determine the time step Δt using Courant-Friedrichs-Lewy criterion $\Delta t = C\frac{\Delta x}{v_{\max}}$ at each time step, where C is the safety factor less than 1 (e.g. C=0.4). To calculate v_{max}, sound speed $c_s = \sqrt{\frac{\gamma p}{\rho}}$ has to be taken into account, so $v_{max}=\max(|v|, c_s)$ over the computational domain.

Close the system of equations by connecting the gas pressure p and total energy density ε with the equation of state given by

$$\varepsilon = \frac{p}{\gamma - 1} + \frac{\rho v_x^2}{2}.$$

Note that the momentum equation can (and should) be solved with respect to ρv_x variable, so that momentum and velocity are connected by $m=\rho v_x$.

Use $\gamma=4/3$. Use domain size x=0..1. As the initial condition, use Sod shock tube:

$$x < 0.5 : \rho = 1; p = 1; v_x = 0$$
$$x \geq 0.5 : \rho = 0.125; p = 0.1; v_x = 0$$

Periodic boundary conditions can still be used, however, it is better to use constant derivative boundary conditions, so d/dx at the boundary cell is equal to d/dx at the nearest inner domain cell. The best would be to make a switch and be able to select the boundary condition type.

Plot the evolution of the solution (density, velocity, internal energy, pressure) on time. Your solution should be numerically stable. After a number of timesteps, you should get a distinct shape with rarefaction wave, contact discontinuity and shock discontinuity (shown to the right). Note also, due to high diffusivity of the scheme, a large number of grid cells (N~1000) are beneficial to use.