Econ 1202/2291

Quantitative Methods A

A Survey of Lecture Examples

School of Economics
University of New South Wales

WARNING!

The following examples were worked out in the lectures over the past session. This gives rise to three implications:

1. These were given as part of a *Lecture*, hence, to look at them outside of their original context is *dangerous*. Be sure to understand where these were coming from in the context of the lecture material;

2. Since they are *Lecture* examples, they are subject to the constraints of lectures: they do not aim to 'cover' all the techniques that you are expected to master in the course, instead, they are necessarily *exemplorary* in nature.

3. Lecture examples also intend to demonstrate a *particular* application of theory, and so, you will not be testing your ability to recognise *which* kind of question you have before you when reading these through (they are often labelled!). Life does not usually come in such neat packages. Therefore, be careful to study 'unseen' questions – ones that you do not know the context of, and hence, have to figure out on the spot how to solve.

A thorough prepartion for your mid- and final- examination will be best served by going through the set Tutorial questions, these are far more exhaustive of the course material.

Lecture 1

1.1

Q: The world is experiencing exponential growth in population, but declining economic stocks of energy, fresh water and food. Solve.

A: (Left as an exercise for the student.)

1.2

Q: *¡title¿* ¡problem¿

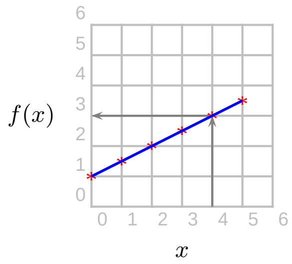
A: ¡solution¿

Lecture 2

2.1

Q: *A linear function* Consider what is meant by the simple linear function $f(x) = 1 + 0.5x$.

A: "*ef-of-x* is equal to 1 plus 0.5 times x." Each x value put into function f will give **exactly one** $f(x)$ value. If we are dealing with CONTINUOUS inputs (x can take **any** value between 0 and 5 say), then we normally draw a line to represent the functional relationship between $f(x)$ and x.



2.2

Q: *Dependent, Independent* Identify the dependent and independent terms, and the value and argument of the function $y = H(a,b,c) = a^2 + 2b + 3$.
A:

- y **depends** on a and b (the **independent** variables);
- y is the **value**, while a and b are the **arguments**.
- H is the **function name**.

2.3

Q: Find the domain of the function, $y(x) = \frac{2}{x^2+3x-4}$.
A: Factorising gives,

$$y(x) = \frac{2}{(x-1)(x+4)} \ ,$$

which implies, $x \neq 1$ or -4. Hence, the **domain** of y is the Real numbers **except** 1, -4.

2.4

Q: *Combining functions* Suppose $f(x) = 2x^2 - 3x - 2$ and $g(x) = x - 2$, and let $h(x) = \frac{f}{g}(x)$, then show that $(h-g)(x) = x + 3$.
A: We begin by finding $h(x)$,

$$h(x) = \frac{f}{g}(x) = \frac{2x^2-3x-2}{x-2} = \frac{(2x+1)(x-2)}{x-2} = 2x+1 \ ,$$

now,

$$\begin{aligned} (h-g)(x) &= h(x) - g(x) \\ &= 2x + 1 - (x-2) = x + 3 \ . \end{aligned}$$

2.5

Q: *Composite functions* Let $p(x) = x^2 - 2$, and $h(x) = \sqrt{5x+1}$ (for $x \geq 0$). Find $(p \circ h)(2)$.
A: We have,

$$(p \circ h) = p(h(x))$$

solving **by substitution**,

$$\begin{aligned} p(h) &= h^2 - 2 \\ \therefore \quad p((h(x)) &= (\sqrt{5x+1})^2 - 2 \\ \therefore \quad p((h(2)) &= (5)(2) - 1 = 9 \end{aligned}$$

2.6

Q: Suppose $f(x) = \frac{x^2+1}{5}$, find $f^{-1}(x)$.
A: First, let

$$y = f(x) = \frac{x^2+1}{5} \ ,$$

now, solve for x **in terms of** y,

$$\begin{aligned} \therefore \ 5y - 1 &= x^2 \\ \therefore \ x &= \sqrt{5y-1} \end{aligned}$$

$$f^{-1}(x) = \sqrt{5x-1}$$

Lecture 3

3.1

Q: *Simple Interest* Suppose we invest (as in our case), $P = \$1500$, $r = 5\%$p.a and $t = 2$yrs. How much interest (calculated simply) would we gain? What would be the final value of our investment?
A: In the first case we have,

$$\begin{aligned} I &= Prt \\ \therefore \ I &= (1500)(0.05)(2) \\ &= \$150 \end{aligned}$$

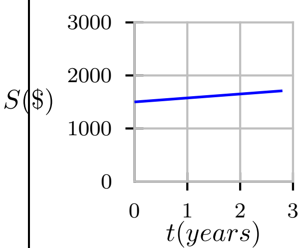
which means that our final value would be,

$$\begin{aligned} S &= P + I \\ \therefore \ S &= 1500 + 150 \\ &= \$1,650 \end{aligned}$$

3.2

Q: *Reverse simple interest* With the same interest rate as in the previous example, how many years would it take to gain our target ($3,200)?
A: We solve for t,

$$\begin{aligned} S &= P + I = P + Prt \\ f(x) &= (P) + (Pr)x = c + mx \\ \therefore \ t &= \frac{S-P}{Pr} \\ &= \frac{3200-1500}{(1500)(0.05)} \\ &\simeq 22 \text{ years, } 8 \text{ months !} \end{aligned}$$



3.3

Q: *Compund Interest* Find the final value of a $1000 investment, invested for 5 years at the nominal rate of 8% compounded quarterly.
A: First, find the **periodic rate of interest**: since the period is quarterly, this means,

$$r = \frac{0.08}{4} = 0.02$$

Next, find the total number of periods, $n = 5 \times 4 = 20$; solve,

$$\begin{aligned} S &= P(1+r)^n \\ &= 1000(1+0.02)^{20} \\ &= \$1,485.95 \end{aligned}$$

3.4

Q: *Option 2: reverse compound interest* Returning to our problem, how long would it take to obtain \$3,200 with an initial investment of \$1,500, if interest is compounded every **two months** at a nominal rate of 5%?
A: We have,

$$S = P(1+r)^n$$

rearranging (using our log laws!),

$$\begin{aligned} \frac{S}{P} &= (1+r)^n \\ \therefore \quad \ln(S/P) &= n\ln(1+r) \\ \therefore \quad n &= \frac{\ln(S/P)}{\ln(1+r)} \end{aligned}$$

So, correcting the interest rate, we obtain,

$$\begin{aligned} n &= \frac{\ln(\frac{3200}{1500})}{\ln(1+\frac{0.05}{6})} \\ &= 91.3 \textbf{ periods} \\ &= 91.3/6 \textbf{ years} \\ &\simeq 15 \text{ years, 2 months} \end{aligned}$$

3.5

Q: *Option 3: Continuous compounding* With $P = \$1500$ and $r = 0.05$, after two years, what will be the value of the investment under continuous compounding?
A: Applying formula (**??**),

$$\begin{aligned} S &= (1500)e^{(0.05)(2)} \\ &= \$1,657.76 \end{aligned}$$

Check with (**??**) our original formula, supposing **daily** compounding,

$$\begin{aligned} S &= P(1+r)^n \\ &= (1500)(1+\frac{0.05}{365})^{2\times 365} \\ &= \$1,657.74 \quad ! \end{aligned}$$

3.6

Q: *for the record...* With **continuous compounding**, initial investment of \$1,500 and annual interest rate of 0.05, how long would it take to obtain a total investment of \$3,200?
A: We have,

$$S = Pe^{rt}$$

Rearranging,

$$\begin{aligned} \frac{S}{P} &= e^{rt} \\ \therefore \quad \ln\frac{S}{P} &= rt \qquad \text{recall: } \ln(e^a) = a \\ \therefore \quad t &= \frac{1}{r}\ln\frac{S}{P} \end{aligned}$$

So, by substitution,

$$\begin{aligned} t &= \frac{1}{0.05}\ln\frac{3200}{1500} \\ &\simeq 15 \text{ years, 2 months} \end{aligned}$$

Lecture 4

4.1

Q: A bank is offering two different types of savings accounts – the first has a nominal rate of 5.15% compounded quarterly, whilst the other just has a single yearly compounding at a rate of 5.35%. Which would you choose?
A: Applying (**??**) to the first account,

$$r_e = (1+\frac{0.0515}{4})^4 - 1 = 5.25\%$$

Checking (with \$1):

$$\begin{aligned} S &= P(1+\frac{r}{n})^{nt} \\ &= (1)(1+\frac{0.0515}{4})^{4\times 10} \\ &= \$1.67 \end{aligned}$$

or using r_e,

$$\begin{aligned} S &= P(1+r_e)^t \\ &= (1)(1+0.0525)^{10} \\ &= \$1.67 \end{aligned}$$

4.2

Q: *Approach 1: just their bank account* Suppose that the money is put into a very safe, conservative bank account which guarantees a 3% per annum single yearly compounded return.
A: Working backwards, we have,

$$S = P(1+r)^n$$

which gives,

$$P = \frac{S}{(1+r)^n} = S(1+r)^{-n}$$

substituting,

$$\begin{aligned} P &= 100,000(1+0.03)^{-20} \\ &= \$55,367.58 \quad \text{(still a lot!)} \end{aligned}$$

4.3

Q: *Approach 2: their super fund* Now suppose they are going to use money from their superannuation fund (which they fortuitously can't touch for 20 years anyhow). Suppose that it receives no more outside money between now and 20 years into the future, just the interest it yields, which is calculated semi-annualy at around 11% per annum (it's a good fund). How much would

need to be in there now?
A: Again, but just with a little more care,

$$\begin{aligned} P &= S(1+r)^{-n} \\ &= 100,000(1+\frac{0.11}{2})^{-(20\times 2)} \\ &= \$11,746.31 \quad ! \end{aligned}$$

4.4

Q: *Approach 3: the 'Endowment Trust' fund* Suppose instead, that the money will come from a dedicated fund for this purpose offered by their bank. This account uses **continuous compounding** at a nominal rate of 11%. How much would need to be invested now?
A: From last time, we have,

$$S = Pe^{rt}$$

rearranging, and solving,

$$\begin{aligned} P &= (S)/(e^{rt}) \\ &= Se^{-rt} \\ \therefore \quad P &= 100,000e^{-0.11\times 20} \\ &= \$11,080.32 \end{aligned}$$

4.5

Q: *Dummy check case study: A can of Coke* If a can of (regular) Coca-cola costs \$1.80 today, what would it have cost me when I was a kid (20 years ago)? (Assume 3% inflation through these years).
A:

$$\begin{aligned} P &= S(1+r)^{-n} \\ &= (1.80)(1+0.03)^{-20} \\ &\simeq \$1.00 \end{aligned}$$

4.6

Q: Using a focal date of **now** or in **12 months**, and simple interest at the nominal value of 7%, what would be the single sum you owe?
A: Let x be the payment now, then,

$$\begin{aligned} \text{value of repayment} &= \text{value of debts} \\ \therefore \quad x + 100(1+0.07\frac{0}{12}) &= 500(1+0.07\frac{6}{12})^{-1} + 350(1+0.07\frac{9}{12})^{-1} \\ \therefore \quad x + 100 &= 483.09 + 332.54 \\ \therefore \quad x &= \$715.63 \end{aligned}$$

Let y be the payment in 12 months, then,

$$\begin{aligned} \text{value of repayment} &= \text{value of debts} \\ \therefore \quad y + 100(1+0.07\frac{12}{12}) &= 500(1+0.07\frac{6}{12}) + 350(1+0.07\frac{3}{12}) \\ \therefore \quad y + 107 &= \$517.5 + 356.13 \\ \therefore \quad y &= \$766.63 \end{aligned}$$

Lecture 5

5.1

Q: Suppose your friend has access to a bank who has a long-term savings account (yearly compounded) offering a nominal rate of 12%. Should *The Bean House* get off the ground?

Year ending	Costs	Income	I - C	$(1+0.12)^{-t}$	PV
0	50,000	0	-50,000		
1	34,000	25,000	-9,000		
2	34,000	45,000	11,000		
3	44,000	60,000	16,000		
4	44,000	70,000	26,000		
5	44,000	75,000	31,000		
				NPV	

A:

$$PV = S(1+r)^{-t} = (I-C)(1+0.12)^{-t}$$

Year ending	Costs	Income	I - C	$(1+0.12)^{-t}$
0	50,000	0	-50,000	1.000
1	34,000	25,000	-9,000	0.893
2	34,000	45,000	11,000	0.797
3	44,000	60,000	16,000	0.712
4	44,000	70,000	26,000	0.636
5	44,000	75,000	31,000	0.567

Year end	Costs	Income	I - C	$(1+0.12)^{-t}$	PV
0	50,000	0	-50,000	1.000	$-50,000$
1	34,000	25,000	-9,000	0.893	$-8,036$
2	34,000	45,000	11,000	0.797	$8,769$
3	44,000	60,000	16,000	0.712	$11,388$
4	44,000	70,000	26,000	0.636	$16,523$
5	44,000	75,000	31,000	0.567	$17,590$
				NPV	$-3,764$

5.2

Q: *IRR by hand* Suppose a project requires an initial investment of \$20,000 and returns \$7,000 and \$16,000 at the end of the first and second years respectively. Find the IRR of the project assuming yearly compounding.
A:

$$NPV = -20,000 + 7,000(1+r)^{-1} + 16,000(1+r)^{-2}$$

Set $NPV = 0$ and solve for r. Multiply through by $(1+r)^2$:

$$\begin{aligned} 20,000(1+r)^2 &= 7,000(1+r) + 16,000 \\ \therefore \quad 20,000 + 40,000r + 20,000r^2 &= 7,000 + 7,000r + 16,000 \\ \therefore \quad 20,000r^2 + 33,000r - 3,000 &= 0 \end{aligned}$$

Solve with the quadratic formula:

$$\begin{aligned} r &= \frac{-33,000 \pm \sqrt{33,000^2 - (4)(20,000)(-3,000)}}{(2)(20,000)} \\ \therefore \quad r &= 0.0864, -1.736 \quad \text{so...} \quad r^* = 8.64\% \end{aligned}$$

5.3

Q: Using a computer program of your choice, find the *IRR* and *NPV* (at 5%) of the following stream of net profits: (-45, -25, -2, 12, 27, 30, 31).
A: Going to my program, I enter the data into one column (cells A1:A7). Then, I use the function IRR:

```
=IRR(A1:A7)
```

Giving: 7.81%.Now, for the *NPV* care is required, the syntax is:

```
=NPV(r,F1,F2,...,FT)
```

Note: no F0 – that's right, to get the correct outcome, you would have to write (in our case):

```
=NPV(0.05,A2:A7)+A1
```

Giving: $9. Try it with $r = 0.0781$.

Lecture 6

6.1

Q: Find the sum of the sequence

$$2, 4, 8, 16, 32, 64, 128, 256, 512$$

A: Applying formula (??), with $a = 2$, $b = 2$ and $n = 9$,

$$\begin{aligned} s &= \frac{2(1-2^9)}{1-2} \\ &= \frac{2(1-512)}{-1} \\ &= 1022 \end{aligned}$$

Check (by hand): $2+4+8+16+32 = 62$, $64+128+256+512 = 960$: $960 + 62 = 1022$.

6.2

Q: *Present Value of Annuity* Show that the present value A of an annuity of agreed payments R, paid at the end of each of n periods, with r interest rate (per period), is given by (hint: use the geometric progression result in (??)),

$$A = R \cdot \frac{1-(1+r)^{-n}}{r} \ .$$

A: Writing out the present value of the annuity components we have,

$$A = R(1+r)^{-1} + R(1+r)^{-2} + \cdots + R(1+r)^{-n} \ ,$$

which is just the same as a **geometric series** with initial value $R(1+r)^{-1}$, and constant factor $(1+r)^{-1}$, using our formula (??),

$$\begin{aligned} A &= \frac{R(1+r)^{-1}\left[1-(1+r)^{-n}\right]}{1-(1+r)^{-1}} \\ &= \frac{R\left[1-(1+r)^{-n}\right]}{(1+r)\left[1-(1+r)^{-1}\right]} = \frac{R\left[1-(1+r)^{-n}\right]}{(1+r)-1} \\ &= R \cdot \frac{1-(1+r)^{-n}}{r} \ . \end{aligned}$$

6.3

Q: *Back to the Mobile-phone* In our scenario, you are making $20 payments on your mobile phone every month for two years. What is the real cost (present value) of the phone? (Assume the company charges 8% interest.)
A: First, we identify the information: the repayment period is **one month** so the **interest rate per period** is $0.08/12 = 0.0067$; also, the number of periods is $2 \times 12 = 24$. Next we use formula (??),

$$\begin{aligned} A &= R \cdot \frac{1-(1+r)^{-n}}{r} \\ &= 20 \cdot \frac{1-(1+0.0067)^{-24}}{0.0067} \\ &= \$442.21 \text{ .(try your own phone...)} \end{aligned}$$

6.4

Q: Calculate the previous example with a spreadsheeting program, and check the result. Set it up so that you can alter the interest rate. What happens? (Why?)
A: There's no easy formula, unfortunately. So...

Period (n)	Payment	PV factor $(1+r)^{-n}$	PV Payment
1	$20	0.993	19.86
2	$20	0.987	19.74
3	$20	0.980	19.60
...	...	...	...
24	$20	0.853	17.06
		Sum	$442.21

6.5

Q: *Car loan* Suppose you are considering purchasing a car. The model you want will cost you $13,450. You have access to a loan through your bank, who would charge 9.50% interest. What would the monthly repayments be if the term of the loan was 5 years?
A: What we have, is an annuity, but this time we know the **present value** ($13,450) and just have to find the **fixed repayments** that give this amount. So, rearranging,

$$\begin{aligned} A &= R \cdot \frac{1-(1+r)^{-n}}{r} \\ \therefore \quad R &= A \cdot \frac{r}{1-(1+r)^{-n}} \end{aligned}$$

substituting in our information, (periodic rate $0.095/12 = 0.0079$)

$$\begin{aligned} R &= 13,450 \cdot \frac{0.0079}{1-(1+0.0079)^{-5\times 12}} \\ &= \$282.48 \quad \text{per month} \ . \end{aligned}$$

6.6

Q: Suppose you put $100 on the first of each month into a savings account which pays 5.5% accumulated monthly. What would be size of the account after 45 years?

A: Using formula (**??**) for the future value of an annuity due, we have (with monthly rate $0.055/12 = 0.0046..$,

$$\begin{aligned} S &= (1+r)\left[R\cdot\frac{(1+r)^n-1}{r}\right] \\ &= (1+0.0046..)\left[100\cdot\frac{(1+0.0046..)^{45\times 12}-1}{0.0046..}\right] \\ &= \$237,038.79 \end{aligned}$$

Lecture 7

7.1

Q: Suppose you gain access to a bank that is offering a 7.5% interest (compounded monthly) savings account which you can deposit a fixed amount into every month. If you need the account to contain \$6,500 in 4 years' time, what would the monthly (end of month) deposits be?

A: We recognise this as a case of an **ordinary annuity**, with final value \$6,500, periodic rate $0.075/12 = 0.00625$, therefore,

$$\begin{aligned} S &= R\cdot\frac{(1+r)^n-1}{r} \\ \therefore\quad R &= S\cdot\frac{r}{(1+r)^n-1} \end{aligned}$$

substituting our numbers,

$$\begin{aligned} R &= 6500\cdot\frac{0.00625}{(1+0.00625)^{4\times 12}-1} \\ &= \$116.54 \text{ (only!)}\ . \end{aligned}$$

7.2

Q: Suppose that instead of putting the deposits into the bank at the end of each period (as in the last example), you deposit at the beginning of the period. Given the same data as the previous example, what would be the monthly deposits now?

A: This time, we're dealing with an **annuity due**, together with the data as before (so $r = 0.00625$, and $n = 4\times 12$),

$$\begin{aligned} S &= (1+r)\left[R\cdot\frac{(1+r)^n-1}{r}\right] \\ \therefore\quad R &= \frac{S}{1+r}\left[\frac{r}{(1+r)^n-1}\right] \end{aligned}$$

substituting our numbers,

$$\begin{aligned} R &= \frac{6500}{1+0.00625}\left[\frac{0.00625}{(1+0.00625)^{4\times 12}-1}\right] \\ &= \$115.81 \qquad \text{(why is it lower?)} \end{aligned}$$

7.3

Q: You've gone a bit over-board in buying your first car, and are locked into a monthly repayment of \$420 (at nominal rate 9.6%) for the next 5 years. You are thinking to yourself, if at the end of your degree (in 3 years' time) you get a 'good job' you figure you may be able to pay it off in one hit. How much will be left on the loan?

A: With the normal translations of the information, and assuming end of period calculation (to get the first pay-cheque), we apply formula (**??**),

$$\begin{aligned} P_k &= 420\cdot\frac{1-(1+0.008)^{-(60-36)}}{0.008} \\ &= \$9,138.28 \quad (P_0 = \$19,952) \end{aligned}$$

7.4

Q: Back to the expensive car from the previous example. What's the interest and principal component of the payment at the end of the 36th period? If the loan was paid out in normal time, what would be the total interest paid?

A: We just apply our formulae,

$$\begin{aligned} I_k &= rP_k \\ &= 0.008\times 9,138 = \$73.10 \end{aligned}$$

$$\begin{aligned} p_k &= R - I_k \\ &= 420 - 73 = \$346.90 \end{aligned}$$

Total interest,

$$\begin{aligned} I &= nR - A \\ &= (5\times 12)420 - 19,952 \\ &= \$5,248 \end{aligned}$$

Lecture 8

8.1

Q: Suppose your bank offers you a savings account which is compounded monthly at a nominal rate of 11.5%. You plan to transfer \$2200 into the account every quarter. How much would be in the account after 6 years?

A: Let r be quarterly periodic rate, then,

$$\begin{aligned} (1+r)^4 &= (1+0.115/12)^{12} \\ \therefore\quad 1+r &= (1.0096)^{\frac{12}{4}} \\ \therefore\quad r &= (1.0096)^{\frac{12}{4}} - 1 \\ \therefore\quad r &= 0.02907...\ . \end{aligned}$$

Now, do simple FV annuity calculation,

$$\begin{aligned} S &= 2200\cdot\frac{(1+0.029...)^{6\times 4}-1}{0.029...} \\ &= \$74,868.80\ . \end{aligned}$$

8.2

Q: *Challenging* How often in a year must I place \$10 in an account which yields 12% compounded semi-annually, such that by the end of the year it has \$1,300 in it?

A: First work on the general annuity problem, then we rearrange the future value formula and solve for n:

$$\begin{aligned}(1+r)^n &= \left(1+\frac{0.12}{2}\right)^2 \\ r &= (1+0.06)^{\frac{2}{n}}-1 \\ &= (1.06)^{\frac{2}{n}}-1\end{aligned}$$

$$\begin{aligned}S &= R\cdot\frac{(1+r)^n-1}{r} \\ \therefore\quad 1300 &= 10\cdot\frac{(1+[(1.06)^{\frac{2}{n}}-1])^n-1}{(1.06)^{\frac{2}{n}}-1}\end{aligned}$$

$\therefore\quad \frac{1300}{10}[(1.06)^{\frac{2}{n}}-1]=(1.06)^2-1$

$\therefore\quad n=\frac{2\log 1.06}{\log\left(\frac{1.06^2-1}{130}+1\right)}$

$\therefore\quad n=$ 122.6 periods, or about every 3 days

8.3

Q: Suppose you plan to put \$1350 at the end of twelve months time into an account earning 7.8% compounded quarterly, and then every 6 months after that. What would be the value of the account in 4 years time (from now)?

A: First, we need to change the rate since we have a general annuity:

$$\begin{aligned}(1+r)^2 &= \left(1+\frac{0.078}{4}\right)^4 \\ r &= \left(1+\frac{0.078}{4}\right)^{\frac{4}{2}}-1 \\ &= 0.03938...\end{aligned}$$

Then, work with the deferred annuity (deferred for just one period), where the total periods is $4\times 2-1=7$,

$$\begin{aligned}S_k &= 1350\cdot\frac{(1+0.039...)^7-1}{0.039...} \\ &= \$10,642.66\ .\end{aligned}$$

8.4

Q: Suppose you are planning to place \$1000 per year into a yearly compounded savings account, at 8% interest. What would be the present value if the term were 10 years? 30 years? 50 years?

A: Applying our PV to an annuity formula three times,

$$A=R\cdot\frac{1-(1+r)^{-n}}{r}\ ,$$

yields,

$$\begin{aligned}PV_{10} &= \$6,710.08 \\ PV_{30} &= \$11,257.78 \\ PV_{50} &= \$12,233.48\end{aligned}$$

8.5

Q: *A perpetuity problem* BHP is deciding between two ways of supporting four scholarships a year in the School of Economics: one being to pay the scholarship each year direct to the students, or else they are considering setting up a trust to fund the scholarships in perpetuity. Supposing that the going interest rate is 11% for the perpetuity, and their alternative cash account attracts 3% and that each scholarship is to be worth \$ 17,900 per year, how long does the scholarship have to last for the perpetuity to be a better option for them?

A:

$$A=\frac{R}{r}=\frac{(4)(17,900)}{0.11}=\$650,909.09$$

$$\begin{aligned}A &\le R.\left(\frac{1-(1+r)^{-n}}{r}\right) \\ \therefore\quad \frac{Ar}{R} &\le 1-(1+r)^{-n} \\ \therefore\quad (1+r)^{-n} &\le 1-\frac{Ar}{R} \\ \therefore\quad -n\log(1+r) &\le \log\left(1-\frac{Ar}{R}\right) \\ \therefore\quad n &\ge -\frac{\log\left(1-\frac{Ar}{R}\right)}{\log(1+r)} \\ \therefore\quad n &\approx 11 years\end{aligned}$$

8.6

Q: *¡title¿* ¡problem¿

A: ¡solution¿

Lecture 9

9.1

Q: *Straight-line method* Suppose that you want to put four year-old bike (which originally cost you \$950) onto `tradingpost.com` and are trying to determine what price you should attach to it. You decide on the **straight-line method** of depreciation, figuring that the bike's total life will be around 15 years, at which time it will be worth \$50 for parts. What is it now worth?. What is the loss in value if you delay the sale one year?

A: Identifying the information: original cost = \$950, scrap value = \$50, total-life = 15 years, current year = 4.

$$\begin{aligned}B_4=C-4\left(\frac{C-S}{n}\right) &= 950-4\frac{950-50}{15} \\ &= \$710\ .\end{aligned}$$

By delaying, we just lose one more year's worth of depreciation, or,

$$\begin{aligned}\frac{C-S}{n} &= \frac{950-50}{15} \\ &= \$60\ .\end{aligned}$$

9.2

Q: *Constant percentage method* Assuming the constant percentage method of depreciation, find an expression for the number of years it would take for an asset to lose half its value. Then, for 5%, 10% and 15% rates of depreciation, calculate an asset's 'half-life'.

A: Let $B_k=\frac{1}{2}C$, then,

$$
\begin{aligned}
B_k = \frac{1}{2}C &= C(1-d)^k \\
\therefore \quad \log\frac{1}{2} &= k\log(1-d) \\
\therefore \quad k_{\frac{1}{2}} &= \frac{\log\frac{1}{2}}{\log(1-d)} \, .
\end{aligned}
$$

Now, plug in 0.05, 0.10, 0.15,

$$
\begin{aligned}
k_{\frac{1}{2}}(0.05) &= 13.5 \text{ years} \, , \\
k_{\frac{1}{2}}(0.10) &= 6.6 \text{ years} \, , \\
k_{\frac{1}{2}}(0.15) &= 4.3 \text{ years} \, .
\end{aligned}
$$

9.3

Q: Suppose you have purchased a car for \$13,500, and estimate it to have a useful life of 12 years, at which point it will be worth \$500 in scrap metal. Using a sinking fund method, prepare a **depreciation schedule** at 12% interest compounded annually.
A: First, we set the future value of the fund to be equal to the depreciation base, $13,500 - 500 = \$13,000$.

Next we work out the annual periodic payments required such that the future value of the fund will equal the depreciation base,

$$
\begin{aligned}
R &= S \cdot \frac{r}{(1+r)^n - 1} \\
&= 13000 \cdot \frac{0.12}{(1+0.12)^{12} - 1} \\
&= \$538.68
\end{aligned}
$$

9.4

Q: *¡title¿* ¡problem¿
A: ¡solution¿

Lecture 10

10.1

Q: *The number-plate population* The first step in dealing with our scenario is to work out the total number of number plates possible of the kind $\{LL \cdot \#\# \cdot LL\}$ where L stands for any captial letter from $A - Z$ and $\#$ stands for any number from 1-9.
A: It is useful to think in terms of **boxes** to fill. In this case, we are allowed repeats, hence,

L	L	#	#	L	L

Each L can always be one of 26 possible different letters, and each $\#$ can be one of 9 possible different numbers, hence,

26	26	9	9	26	26
L	L	#	#	L	L

The trick is that to fix the first 5 of the boxes (say, ‘$AA|37|A\_$’), means there are 26 plates with that fixture; and to fix four boxes, gives 26×26 plates with that fixture ... and so on. Hence,

$$
\begin{aligned}
n(plates) &= 26 \times 26 \times 9 \times 9 \times 26 \times 26 \\
&= 37,015,056
\end{aligned}
$$

10.2

Q: NSW recently changed its plates from the kind $\{LLL \cdot \#\#\#\}$ to $\{LL \cdot \#\# \cdot LL\}$. What's the difference in the number of plates available to the RTA due to the change?
A: Applying our boxes to the first case:

26	26	26	9	9	9
L	L	L	#	#	#

We obtain,

$$
\begin{aligned}
n(\text{plates}_{\text{old}}) &= 26 \times 26 \times 26 \times 9 \times 9 \times 9 \\
&= 26^3 \times 9^3 \\
&= 12,812,904
\end{aligned}
$$

So the change (final-initial) is:

$$
\text{change} = 37,015,056 - 12,812,904 = 24,202,152
$$

(What would the **ratio** be?)

10.3

Q: *For the record...* The police have a photo from the Kingsford BP petrol station that shows ‘AA_3_K’. How many number plates could have this configuration? What proportion of the total number do they comprise?
A: Using either boxes or trees:

$$
\begin{aligned}
n_{\text{AA\_3\_K}} &= 1 \times 1 \times 9 \times 1 \times 26 \times 1 \\
&= 9 \times 26 = 234
\end{aligned}
$$

The proportion of all plates of this kind is then,

$$
\text{Just} \quad \frac{234}{37,015,056} = 0.0000063
$$

10.4

Q: Suppose that number plates are of the (old) form ‘$LLL \cdot \#\#\#$’ and the thief mentioned above is known to have the letters $\{B, D, E, F, J, L, M, P, R, S, U, V\}$ and the numbers $\{2, 5, 6, 7, 8, 9\}$. How many different rear number plates could he come up with?
A: We need to two permutation calculations, of the kind **‘12 objects, choose 3’** and **‘6 objects, choose 3’** respectively. For the letters, the first box has 12 choices, the next box 11 choices, then 10 (recall, we only have one of each letter). So we have,

$$
n(LLL) = 12 \times 11 \times 10 = 1,320
$$

for the numbers, a similar process leads to,

$$
n(\#\#\#) = 6 \times 5 \times 4 = 120
$$

So taken together (using the Basic Counting Principle), we have

$$
n(pates) = 1,320 \times 120 = 158,400(!)
$$

10.5

Q: *On Factorials* Try to do $\frac{112!}{109!}$ on your calculator. Problems? Can you work it out another way??
A: Let's look at the formula:

$$\begin{aligned}\frac{112!}{109!} &= \frac{112 \times 111 \times 110 \times 109 \times \ldots 2 \times 1}{109 \times 108 \times \ldots 2 \times 1} \\ &= 112 \times 111 \times 110 \\ &= 1,367,520\end{aligned}$$

Actually, we got back the same as for,

$$\begin{aligned}{}_{112}P_3 &= \frac{112!}{(112-3)!} \\ &= 112 \times 111 \times 110 \\ &= 1,367,520\end{aligned}$$

Try it on your calculator using $\boxed{{}_nP_r}$. *(But try just 112!)*

10.6

Q: *Back to the thief* Suppose now that there is a glitch in the video recording system, and that when they use the recognition technology, all it tells the police is which **selection** of letters and numbers was used on the fake plate, *not what order they were in.* If the police know that a thief has 15 different lettering decals, how many combinations of (just) the three letters will the thief be able to make?
A: To do by hand wouldn't be much fun! .. but, with $n = 15$ and $r = 3$ we have:

$$\begin{aligned}{}_{15}C_3 &= \frac{15!}{3!(15-3)!} \\ &= \frac{15!}{3!(12!)} \\ &= \frac{15 \times 14 \times 13}{3!} \\ &= 455\ .\end{aligned}$$

(Check with the $\boxed{{}_nC_r}$ *key.)*

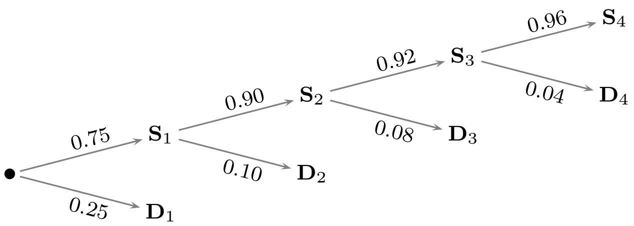
10.7

Q: *¡title¿* ¡problem¿
A: ¡solution¿

Lecture 11

11.1

Q: *Multiplication Law* Suppose that the administration of the University of Coogee have been doing some surveys of the timing of students dropping out of their five-year degree programs. The results indicate that 25% drop out after first year, and then for the subsequent three years, 10%, 8% and 4% of the students who stay each year will drop out. Given these data, what is the probability that a first year student will drop out after 2, 3 and 4 years?
A:



Applying the multiplication law:

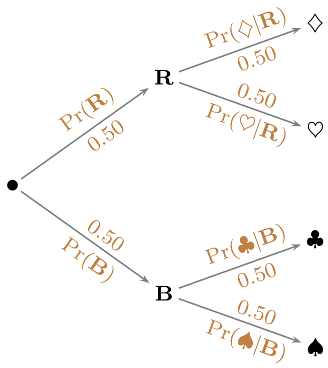
$$\begin{aligned}\Pr(\mathbf{S}_1 \cap \mathbf{D}_2) &= \Pr(\mathbf{S}_1)\Pr(\mathbf{D}_2|\mathbf{S}_1) = (0.75)(0.10) = 0.075 \\ \Pr(\mathbf{S}_2 \cap \mathbf{D}_3) &= \Pr(\mathbf{S}_1 \cap \mathbf{S}_2)\Pr(\mathbf{D}_3|\mathbf{S}_2) = (0.75)(0.90)(0.08) = 0.054 \\ \Pr(\mathbf{S}_3 \cap \mathbf{D}_4) &= \Pr(\mathbf{S}_1 \cap (\mathbf{S}_2 \cap \mathbf{S}_3))\Pr(\mathbf{D}_4|\mathbf{S}_3) = (0.75)[(0.90)(0.92)](0.04)\end{aligned}$$

11.2

Q: *Conditional Probability* Given that a card picked from a full deck is red-suited, what is the probability that it is diamond suited?
A:

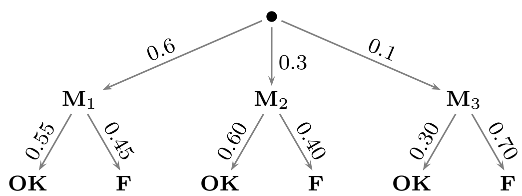
- We have the language of 'given x, what is the probability that y', so we can draw a tree:
- Once we pick a *red* suit, which occurs with $\Pr(\mathbf{R}) = \frac{1}{2}$, we have only **two** options: $\{\diamondsuit, \heartsuit\}$, so we can write the *conditional probability*,

$$\Pr(\diamondsuit|\mathbf{R}) = \frac{1}{2}\ .$$



11.3

Q: *Applying Bayes' Formula* A car manufacturing plant has three car chasis production machines, $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$. For historical reasons, a car rig on the production line has a 0.6, 0.3 and 0.1 probability of going to each machine respectively. If the three machines add a chasis to a rig without fault with 0.55, 0.60 and 0.30 probabilities respectively, what is the probability that a rig with a chasis fault came from $\mathbf{M}_2$?
A:

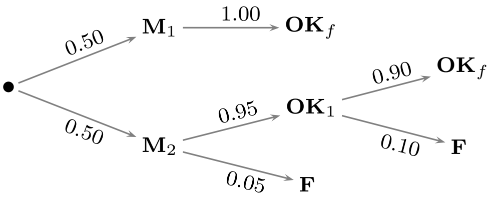


$$\begin{aligned}\Pr(\mathbf{M}_2|\mathbf{F}) &= \frac{\Pr(\mathbf{M}_2)\Pr(\mathbf{F}|\mathbf{M}_2)}{\Pr(\mathbf{M}_1)\Pr(\mathbf{F}|\mathbf{M}_1) + \Pr(\mathbf{M}_2)\Pr(\mathbf{F}|\mathbf{M}_2) + \Pr(\mathbf{M}_3)\Pr(\mathbf{F}|\mathbf{M}_3)} \\ &= \frac{(0.3)(0.40)}{(0.6)(0.45) + (0.3)(0.40) + (0.1)(0.70)} = 0.12/0.46 = 0.26\end{aligned}$$

11.4

Q: *More Bayes' Formula* Suppose that the plant from the previous example restructure their car chasis production line. Now they have just two machines doing 50% of the work each: $\mathbf{M}_1$, which always adds the chasis to the rig without fault, and (the old) $\mathbf{M}_2$, which has had a mid-operation check added to it which is rejecting 5% of the chasis due to early faults, the other 95% go on to the final stage, where 90% of them are faultless. Given that a complete chasis is faultless, what's the probability it came from $\mathbf{M}_2$ now?

A:



$$\begin{aligned}
\Pr(\mathbf{M}_2|\mathbf{OK}_f) &= \frac{\Pr(\mathbf{M}_2)\Pr(\mathbf{OK}_1|\mathbf{M}_2)\Pr(\mathbf{OK}_f|\mathbf{OK}_1)}{\Pr(\mathbf{M}_1)\Pr(\mathbf{OK}_f|\mathbf{M}_1)+\Pr(\mathbf{M}_2)\Pr(\mathbf{OK}_1|\mathbf{M}_2)\Pr(\mathbf{OK}_f|\mathbf{OK}_1)} \\
&= \frac{(0.50)(0.95)(0.90)}{(0.50)(1.00)+(0.50)(0.95)(0.90)} = \frac{0.4275}{0.9275} = 0.46
\end{aligned}$$

11.5

Q: ¡*title*¿ ¡problem¿

A: ¡solution¿

Lecture 12

12.1

Q: Are the matrices

$$\begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

equal?

A: No. They have the same size, but it is not true that each $a_{ij} = b_{ij}$. For example, $a_{21}(2) \neq b_{21}(4)$.

12.2

Q: Determine the solution to,

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

A:

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 4+2 & 9+0 \\ 2+0 & 1+7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}$$

12.3

Q: Determine the solution to,

$$\begin{bmatrix} 4 & 9 & 3 \\ 2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

A: No solution! Sizes are different.

12.4

Q: Let $\mathbf{A}$ be given by,

$$\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix}$$

determine $7\mathbf{A}$.

A: We have,

$$\begin{aligned}
7\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} &= \begin{bmatrix} 7\times 3 & 7\times -1 \\ 7\times 0 & 7\times 5 \end{bmatrix} \\
&= \begin{bmatrix} 21 & -7 \\ 0 & 35 \end{bmatrix}
\end{aligned}$$

12.5

Q: Determine the value of $\mathbf{C} = \mathbf{AB}$ as defined above,

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} .$$

A: Write down the sizes of $\mathbf{A}$ and $\mathbf{B}$,

$$3 \times \cancel{2 \quad \text{and} \quad 2} \times 1$$

check that the **inner dimensions** match and cross them out! Write down the solution matrix size from what remains, 3×1,

$$\mathbf{C} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$$

calculate the sum of **row** times **col** multiplications,

$$\mathbf{C} = \begin{bmatrix} 1(5)+3(9) \\ 2(5)+8(9) \\ 4(5)+0(9) \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}$$

12.6

Q: Under what conditions would the following equation be true?

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$$

A: Let $\mathbf{C} = (\mathbf{A}+\mathbf{B})$. Then the LHS is,

$$\begin{aligned}
\mathbf{C}^2 &= \mathbf{CC} = \mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB} \\
&= (\mathbf{A}+\mathbf{B})\mathbf{A} + (\mathbf{A}+\mathbf{B})\mathbf{B} \\
&= \mathbf{AA}+\mathbf{BA}+\mathbf{AB}+\mathbf{BB} \\
&= \mathbf{A}^2+\mathbf{BA}+\mathbf{AB}+\mathbf{B}^2
\end{aligned}$$

which implies,

$$\begin{aligned}
\mathbf{A}^2+\mathbf{BA}+\mathbf{AB}+\mathbf{B}^2 &= \mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2 \\
\mathbf{BA}+\mathbf{AB} &= 2\mathbf{AB} \quad !!!
\end{aligned}$$

which is only true if $\mathbf{A},\mathbf{B} = \mathbf{I}$ or $\mathbf{A},\mathbf{B} = \mathbf{0}$.

12.7

Q: ¡*title*¿ ¡problem¿

A: ¡solution¿

Lecture 13

13.1

Q: Calculate the determinant of the matrix $\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$.

A: We know that,

$$|\mathbf{A}| = a_{11}a_{22} - a_{21}a_{12} \ .$$

so,

$$\begin{aligned} \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} &= (1)(2) - (5)(3) \\ &= 2 - 15 = -13 \ . \end{aligned}$$

13.2

Q: Find the determinant of the matrix $\begin{bmatrix} 2 & 1 & -2 \\ -5 & 1 & 3 \\ 10 & 2 & 4 \end{bmatrix}$.

A: We proceed as per the definition:

$$\begin{aligned} \begin{vmatrix} 2 & 1 & -2 \\ -5 & 1 & 3 \\ 10 & 2 & 4 \end{vmatrix} &= +(2)\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} - (1)\begin{vmatrix} -5 & 3 \\ 10 & 4 \end{vmatrix} + (-2)\begin{vmatrix} -5 & 1 \\ 10 & 2 \end{vmatrix} \\ &= +(2)[(1)(4) - (2)(3)] \\ &\quad -(1)[(-5)(4) - (10)(3)] \\ &\quad +(-2)[(-5)(2) - (10)(1)] \\ &= -4 + 50 + 40 = 86 \end{aligned}$$

13.3

Q: *Cofactors & Determinants* Find the **cofactor** of the entry '3' in the matrix, $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 1 & 0 & 2 & 1 \end{bmatrix}$

A: The '3' is in **row** 2, **col** 1, so

$$\begin{aligned} a'_{21} &= (-1)^{2+1}\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{vmatrix} \\ &= -\left(+(1)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} - (0)\begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}\right) \\ &= -(-7 + 0) = 7 \end{aligned}$$

13.4

Q: *Finding the adjoint* Find the **adjoint** of the matrix, $\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$.

A: Check for the determinant, $\det(\mathbf{A}) = 4$, so proceed. The cofactors $(a')_{ij}$ are,

$$(a')_{11} = (-1)^2\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \qquad (a')_{12} = (-1)^3\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = -2$$

$$(a')_{13} = (-1)^4\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -4 \qquad (a')_{21} = (-1)^3\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

The others: $(a')_{22} = 1, (a')_{23} = -4, (a')_{31} = -2, (a')_{32} = 2, (a')_{33} = 8$ Construct the matrix $\mathbf{A}' = \begin{bmatrix} 2 & -2 & -4 \\ 1 & 1 & -4 \\ -2 & 2 & 8 \end{bmatrix}$ and so,

$$\text{adj}(\mathbf{A}) = (\mathbf{A}')^\intercal = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ -4 & -4 & 8 \end{bmatrix}$$

13.5

Q: *The inverse by the adjoint method* Find the **inverse** of $\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$ by the **adjoint method**.

A: We already know that $\mathbf{A}$ has a non-zero determinant ($|\mathbf{A}| = 4$), and we have worked out the **adjoint** of $\mathbf{A}$ in the previous example. So,

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{|\mathbf{A}|}\text{adj}(\mathbf{A}) \\ &= \frac{1}{4}\begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ -4 & -4 & 8 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{bmatrix} \end{aligned}$$

13.6

Q: Determine whether the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 9 & 2 \end{bmatrix}$ has an inverse.

A: (It is square, so proceed...) Solve,

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{21}a_{12} \\ \therefore \quad \begin{vmatrix} 1 & 0 \\ 9 & 2 \end{vmatrix} &= (1)(2) - (9)(0) \\ &= 2 \end{aligned}$$

Since $\begin{vmatrix} 1 & 0 \\ 9 & 2 \end{vmatrix} \neq 0$, we conclude that $\mathbf{A}^{-1}$ exists. (actually, it is $\begin{bmatrix} 1 & 0 \\ -4\frac{1}{2} & \frac{1}{2} \end{bmatrix}$)

13.7

Q: Solve the following system of linear equations by the Adjoint Method:

$$\begin{aligned} p_2 - 3p_3 &= -5 \\ 2p_1 + 3p_2 - p_3 &= 7 \\ 4p_1 + 5p_2 - 2p_3 &= 10 \end{aligned}$$

A: We have: $\mathbf{Ax} = \mathbf{b}$, so solve by left-hand multiplying by $\mathbf{A}^{-1}$. First, check $|\mathbf{A}| \neq 0$. Actually, $|\mathbf{A}| = 6$, proceed... Now, **the cofactors** (recall, $\text{adj}(\mathbf{A}) = (\mathbf{A}')^\intercal$):

$$\mathbf{A}' = \begin{bmatrix} -1 & 0 & -2 \\ -13 & 12 & 4 \\ 8 & -6 & -2 \end{bmatrix} \qquad \textit{Check yourself!}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\text{adj}(\mathbf{A}) = \frac{1}{6}\begin{bmatrix} -1 & -13 & 8 \\ 0 & 12 & -6 \\ -2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -1/6 & -13/6 & 4/3 \\ 0 & 2 & -1 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} -1/6 & -13/6 & 4/3 \\ 0 & 2 & -1 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} -5 \\ 7 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Lecture 14

14.1

Q: Suppose $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are all invertible matrices, and

$$[\mathbf{C}^{-1}\mathbf{A} + \mathbf{X}(\mathbf{A}^{-1}\mathbf{B})^{-1}]^{-1} = \mathbf{C} \ ,$$

express $\mathbf{X}$ in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.
A: Let $\mathbf{D} = [\cdot]$, then

$$\begin{aligned}
\mathbf{D}^{-1} &= \mathbf{C} \\
\mathbf{D}\mathbf{D}^{-1} &= \mathbf{D}\mathbf{C} \\
\mathbf{I} &= [\mathbf{C}^{-1}\mathbf{A} + \mathbf{X}(\mathbf{A}^{-1}\mathbf{B})^{-1}]\mathbf{C} \\
\mathbf{I} &= \mathbf{C}^{-1}\mathbf{A}\mathbf{C} + \mathbf{X}(\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C} \\
\mathbf{X}(\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C} &= \mathbf{I} - \mathbf{C}^{-1}\mathbf{A}\mathbf{C} \\
\mathbf{X} &= [\mathbf{I} - \mathbf{C}^{-1}\mathbf{A}\mathbf{C}][(\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}]^{-1} \\
\mathbf{X} &= [\mathbf{I} - \mathbf{C}^{-1}\mathbf{A}\mathbf{C}](\mathbf{A}^{-1}\mathbf{B})\mathbf{C}^{-1}\mathbf{C}^{-1}(\mathbf{A}^{-1}\mathbf{B})
\end{aligned}$$

14.2

Q: Determine whether the following linear system is **consistent**,

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] , .$$

A: Looking carefully at the last row of the augmented matrix, implies,

$$0x_1 + 0x_2 + 0x_3 = 1 \ ,$$

but this is impossible to satisfy for **any** value of $\mathbf{x}$, so we conclude that the system is **inconsistent**.

14.3

Q: Determine whether the following linear system is **consistent** by checking the determinant of $\mathbf{A}$,

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] .$$

A: So,

$$\begin{aligned}
|\mathbf{A}| &= +(1)\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} + (5)\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\
&= (1)(0) + (3)(0) + (5)(0) \\
&= 0
\end{aligned}$$

Since $|\mathbf{A}| = 0$, $\mathbf{A}$ is **singular**, and so, we **cannot** solve the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, hence it is **inconsistent**.

Lecture 15

(... No written examples for this Lecture ...)

Lecture 16

16.1

Q: A new supplier of power has entered the market who is willing supply you with up to 1000 KW per night (instead of 800 KW per night). What happens to your profits?
A:

1. A new constraint:

$$1000 \geq 5x_1 + 14x_2$$

2. Graph it;
3. Check if constraint is still **binding**;
4. Find new optimum.

16.2

Q: Suppose (with our new Electricity restriction of 1000 KW) that the price of falls progressively from \$8.50 to \$7.50. And then, continues to fall progressively by \$1 each month, finally bottoming-out at \$4.50. What would be the optimum production point (x_1, x_2) and profit at each step along the way (8.50,7.50,6.50,5.50,4.50)?
A: We begin just as before, calculating our profit at our optimum point with the new price:

$$\begin{aligned}
\pi(172, 10)[p_1 = 8.50] &= \$1,366.52 \\
\pi(172, 10)[p_1 = 7.50] &= \$1,194.52 \\
\pi(172, 10)[p_1 = 6.50] &= \$1,022.52 \\
\pi(172, 10)[p_1 = 5.50] &= \$850.52 \\
\pi(172, 10)[p_1 = 4.50] &= \$678.52
\end{aligned}$$

Right? WRONG!!

Lecture 17

17.1

Q: Find the limit at 3 of the function,

$$f(x) = \frac{4}{x^2 - 9} \ .$$

A: Factorising,

$$f(x) = \frac{4}{(x-3)(x+3)}$$

but that implies

$$\lim_{x \to 3} \frac{4}{(0)(6)} = \infty \ .$$

No solution! (limits must be finite)

17.2

Q: Show that the derivative of $f(x) = x^2 + x - 4$ is $2x + 1$.
A: Applying our formula,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \ .$$

we need to solve,

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{((x+\Delta x)^2 + (x+\Delta x) - 4) - (x^2 + x - 4)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\Delta x)^2 + \Delta x + 2x\Delta x}{\Delta x} \quad \text{(exercise!)} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(\Delta x + 1 + 2x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \Delta x + 1 + 2x = 2x + 1
\end{aligned}$$

17.3

Q: Find the **tangent** to the function $f(x) = x^2 + x - 4$ at the point $x = 3$.
A: From our previous example, we have the derivative of $f(x)$ is,

$$f'(x) = 2x + 1$$

so the tangent is just the derivative (gradient, tangent) of $f(x)$ at $x = 3$, or,

$$\begin{aligned}
f'(3) &= (2)(3) + 1 \\
&= 7
\end{aligned}$$

(At the point $x = 3$, for every change by 1 in x, $f(x)$ will change by 7.)

17.4

Q: Find the derivative with respect to x of the following functions,

$$f(x) = -6x^{-2} \qquad g(x) = 2(\sqrt{x})^3 \ .$$

A: First,

$$f'(x) = (-6)(-2)x^{-2-1} = 12x^{-3}$$

and

$$\begin{aligned}
g'(x) &= \frac{d}{dx} 2(x)^{\frac{3}{2}} \\
&= (2)(\frac{3}{2})x^{\frac{3}{2}-1} \\
&= 3x^{\frac{1}{2}}
\end{aligned}$$

17.5

Q: *Sum-difference rule* Given the short-run total-cost function,

$$C = Q^3 - 4Q^2 + 10Q + 75$$

find the **marginal cost** function, that is, the limit of the quotient, $\frac{dC}{dQ}$.
A: Since we only have '+' and '-' operations between each component of Q, we can treat them as individual functions,

$$\begin{aligned}
\frac{dC}{dQ} &= \frac{d}{dQ}Q^3 - \frac{d}{dQ}4Q^2 + \frac{d}{dQ}10Q + \frac{d}{dQ}75 \\
\frac{dC}{dQ} &= 3Q^2 - 8Q + 10 \ .
\end{aligned}$$

17.6

Q: *Product rule* If $y = (2x+3)(3x^2)$, find the derivative $y'(x)$.
A: Applying our **product rule** we have,

$$y = f(x)g(x) \quad \text{where } f(x) = 2x + 3,\ g(x) = 3x^2$$

so,

$$\begin{aligned}
y'(x) &= f(x)g'(x) + g(x)f'(x) \\
&= (2x+3)(6x) + (3x^2)(2) \\
&= 18x^2 + 18x \\
&= 18x(x+1)\ .
\end{aligned}$$

17.7

Q: *Quotient rule* Find the derivative of $y(x) = \frac{2x-3}{x+1}$.
A: We recognise we have a quotient, of the form,

$$\begin{aligned}
y'(x) &= \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)} \\
&\text{where} \quad f(x) = 2x - 3,\ g(x) = x + 1
\end{aligned}$$

so,

$$\begin{aligned}
y'(x) &= \frac{(2)(x+1) - (1)(2x-3)}{(x+1)^2} \\
&= \frac{5}{(x+1)^2}\ .
\end{aligned}$$

17.8

Q: *Chain rule (easy)* Given that $z = 3y^2$ and $y = 2x + 5$, find $\frac{dz}{dx}$.
A: We could do a substitution and expand and solve, or...

$$\begin{aligned}
z'(x) &= \frac{dz}{dy}\frac{dy}{dx} \\
&= (6y)(2) \\
&= 12y \\
&= 12(2x+5)\ .
\end{aligned}$$

17.9

Q: *Chain rule (harder)* (Challenge) Find the solution to $\frac{dz}{dx}$ if

$$z(x) = (x^2 + 3x - 2)^{17}\ .$$

A: This could be very long (expanding to the 17th-power)... Or, suppose we define $y = x^2 + 3x - 2$, then we have,

$$z = y^{17}\ ,$$

which we can now use with our **chain-rule**,

$$\begin{aligned}
z'(y) &= \frac{dz}{dy}\frac{dy}{dx} \\
&= [17y^{16}][2x+3] \\
&= 17(x^2 + 3x - 2)^{16}(2x+3)\ . \quad \text{(nifty!)}
\end{aligned}$$

Lecture 18

18.1

Q: Determine whether y is defined explicitly or implicitly in each of the following functions:

$$x^2 - y^3 + 2 = 0, \quad \ln a - (2-x)^2 = y, \quad (y+x)^2 - 2y = 4$$

A: Only the second equation

$$\ln a - (2-x)^2 = y,$$

is defined **explicitly**, since we have y by itself on one side of the equation, the explicit form of y is on the other side.
The first and third equations yield **implicit** functions for y,

$$y = (x^2+2)^{\frac{1}{3}}, \quad y = 1 - x \pm \sqrt{5-2x} \quad \text{check!}$$

18.2

Q: Find $\frac{dy}{dx}$ given that,

$$y^3 + 3x^2 y = 13 \ .$$

A: Apply $\frac{d}{dx}$ to both sides and collect terms,

$$\begin{aligned} \frac{d}{dx}\left[y^3 + 3x^2 y\right] &= \frac{d}{dx}13 \\ \frac{dy}{dx}3y^2 + 6xy + \frac{dy}{dx}3x^2 &= 0 \quad \text{chain, product rules} \\ \frac{dy}{dx}\left[3y^2 + 3x^2\right] + 6xy &= 0 \end{aligned}$$

Solve for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{-6xy}{3y^2+3x^2} = \frac{-2xy}{x^2+y^2} \ .$$

18.3

Q: Show that $f'(x) = e^x$ when $f(x) = e^x$.
A: Applying our definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} \\ &= \lim_{h\to 0}\frac{e^x.e^h - e^x}{h} \\ &= \lim_{h\to 0} e^x \cdot \frac{e^h - 1}{h} \\ &= e^x \qquad \left(\text{since } \lim_{h\to 0}\frac{e^h-1}{h} = 1\right) \end{aligned}$$

18.4

Q: Differentiate the following:

$$\begin{aligned} y_a &= e^{-x} \ , \\ y_b &= x^p e^{ax} \ , \\ y_c &= \sqrt{e^{2x}+x} \ . \end{aligned}$$

A:

$$\begin{aligned} y_a' &= (-1)e^{-x} = -e^{-x} \ ; \\ y_b' &= px^{p-1}.e^{ax} + x^p.ae^{ax} \ ; \\ y_c' &= \frac{1}{2}(2e^{2x}+1)(e^{2x}+x)^{-\frac{1}{2}} \\ &= \frac{2e^{2x}+1}{2\sqrt{e^{2x}+x}} \end{aligned}$$

18.5

Q: Show that if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$, using the result that $e^{\ln x} = x$.
A: Using the given result, let $\ln x = g(x)$, then we have,

$$e^{g(x)} = x$$

taking the derivative w.r.t. x of both sides,

$$g'(x)e^{g(x)} = 1$$

which implies that,

$$g'(x) = \frac{1}{e^{g(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

18.6

Q: Find dy/dt given that $y = t^3 \ln t^2$.
A: We have a product of two terms, so,

$$\begin{aligned} y'(t) &= t^3\frac{d}{dt}\ln t^2 + \ln t^2 \frac{d}{dt}t^3 \ , \\ &= t^3\left(\frac{2t}{t^2}\right) + (\ln t^2)(3t^2) \\ &= 2t^2 + 3t^2(2\ln t) \\ &= 2t^2(1+3\ln t) \ . \end{aligned}$$

18.7

Q: Find the derivative w.r.t. x of $y = \ln\left(\frac{3x}{1+x}\right)$.
A: Proceed as normal, but must use the **quotient** rule to find the numerator,

$$y'(x) = \frac{g'(x)}{g(x)} \quad \text{where} \quad g(x) = \left(\frac{3x}{1+x}\right)$$

So

$$g'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} = \frac{3(1+x)-(1)(3x)}{(1+x)^2} = \frac{3}{(1+x)^2}$$

Together,

$$y'(x) = \frac{3}{(1+x)^2}\cdot\frac{(1+x)}{3x} = \frac{1}{x(1+x)}$$

18.8

Q: If $f(t) = e^{2t^2+4}$, find $f''(t)$.
A: The first derivative is given by,

$$f'(t) = (4t)e^{2t^2+4} = 4te^{2t^2+4} ,$$

and the second derivative is then just,

$$\begin{aligned} f''(t) &= 4t\frac{d}{dt}e^{2t^2+4} + e^{2t^2+4}\frac{d}{dt}4t , \\ &= (4t)(4t)e^{2t^2+4} + e^{2t^2+4}(4) \\ &= 4e^{2t^2+4}(4t^2+1) . \end{aligned}$$

18.9

Q: For a demand function $q(p)$, explain in words what is meant by the definitions of elastic, unit elastic, inelastic and completely inelastic at some point (q, p).
A: Elastic ($|\eta| > 1$): The demand for the good changes by more than 1% for a given 1% change in the price;
Unit Elastic ($|\eta| = 1$): The demand for the good changes by exactly 1% for a given 1% change in the price;
Inelastic ($|\eta| < 1$): The demand for the good changes by less than 1% for a given 1% change in the price;
Completely inelastic ($|\eta| = 0$): The demand for the good is unchanged for a given 1% change in the price.

18.10

Q: If the demand for coffee is given by the demand equation, $q(p) = 8000p^{-1.5}$, find an expression for the elasticity of demand, and so, approximate the change in quantity demanded for a 1% increase in price at $p = 4$.
A: We have,

$$\eta(q) = \frac{p}{q}\frac{dq}{dp}$$

so,

$$\begin{aligned} \eta(q) &= \frac{p}{8000p^{-1.5}}(8000)(-1.5)p^{-2.5} \\ &= -1.5p^{(1+1.5-2.5)} = -1.5 \end{aligned}$$

So in this case, it turns out that the point elasticity is **independent** of the point we choose – it is always -1.5. Hence, for a 1% increase in the price, we should expect to see a 1.5% decrease in quantity demanded. (How would you find the exact change?)

Lecture 19

19.1

Q: Find all the critical points of the function,

$$c(t) = \frac{t}{t^2+4} .$$

A: Obtain the first derivative,

$$\begin{aligned} c'(t) &= \frac{(1)(t^2+4)-(t)(2t)}{(t^2+4)^2} = \frac{4-t^2}{(t^2+4)^2} \\ &= \frac{(2+t)(2-t)}{(t^2+4)^2} \end{aligned}$$

set $c'(t) = 0$, solve,

$$\begin{aligned} c'(t) &= \frac{(2+t)(2-t)}{(t^2+4)^2} = 0 \\ \therefore \quad t^* &= -2, 2 \end{aligned}$$

The critical values are -2 and 2, giving critical points: (-2,-2/8),(2,2/8).

19.2

Q: Find the nature of the critical values of the function,

$$y = f(x) = x^3 - 12x^2 + 36x + 8 .$$

A: Find the derivative,

$$f'(x) = 3x^2 - 24x + 36 ,$$

find the **critical values** of x by setting $f'(x) = 0$,

$$\begin{aligned} 3x^2 - 24x + 36 &= 0 \\ 3(x-2)(x-6) &= 0 \\ x^* &= 2, 6 \end{aligned}$$

In the neighbourhood of $x = 2$, we find that $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$. Hence, $f(2) = 40$ is a **relative maximum**. Similar logic gives $f(6) = 8$ to be a **relative minimum**.

19.3

Q: Classify the stationary points of the function $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$ by using the second derivative test.
A: First, find $f'(x)$,

$$f'(x) = \frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3} = \frac{1}{3}(x+1)(x-2) ,$$

which gives two stationary points (by setting $f'(x) = 0$) at $x = -1, 2$. Now find $f''(x)$,

$$f''(x) = \frac{2}{3}x - \frac{1}{3} ,$$

which gives $f''(-1) = -1$, and $f''(2) = 1$. Hence, we have a local maximum at $x = -1$ and a local minimum at $x = 2$.

19.4

Q: Determine whether the function $f(x) = x^6$ has an inflection point at $x_0 = 0$.
A: First we check that the necessary condition of $f''(x_0) = 0$ is met,

$$f'(x) = 6x^5 \quad \ldots \quad f''(x) = 30x^4 .$$

So, $f''(0) = 0$. However ... checking that the sign changes ...

$$f''(<0) \to \text{positive} \quad \ldots \quad f''(>0) \to \text{postive} \quad (!!!)$$

In fact, $f''(x) \geq 0$ for all x, so we conclude that $x_0 = 0$ is not an inflection point for f. (It is a *minimum*.)

19.5

Q: Given the data as above, find the profit maximizing output of , and the profits obtained at this point.
A: First, find $\pi'(q)$ and then solve $\pi'(q) = 0$ for q^*:

$$\begin{aligned} \pi(q) &= q - 2q^2 - 148 \\ \therefore \quad \pi'(q) &= -4q \end{aligned}$$

So, solving we have,

$$q^* = /4 = \ .$$

Now, check (by second derivative condition) what kind of extrema we have,

$$\pi''(q) = -4 \quad \text{(negative for all q)}$$

... we have a local maximum. Substituting,

$$\pi() = () - 2(^2) - 148 = \$.$$

Lecture 20

20.1

Q: Suppose as in our example, we have,

$$\frac{dP}{dt} = 0.8t^{-\frac{1}{2}} \ ,$$

find an expression for $P(t)$, the population of Australia each year, where $t = 0$ is the year 2000.
A: What we would like to do, is to find some equation $P(t)$, such that,

$$\frac{d}{dt}P(t) = 0.8t^{-\frac{1}{2}} \ ,$$

... without any other information, we can make a guess:

$$P(t) = (0.8)(2)t^{\frac{1}{2}} \quad \ldots \quad \frac{d}{dt}P(t) = 0.8t^{-\frac{1}{2}}$$

Good! But what if $t = 0$ (that is, the year 2000), *then* $P(t) = 0$*!!*

20.2

Q: Using the same population example as before, find $P(t)$ if $p(t) = 0.8t^{-\frac{1}{2}}$ and $P(0) = 19.6$.
A: From before we guessed an expression for $P(t)$, writing this with C added, we have,

$$P(t) = (0.8)(2)t^{\frac{1}{2}} + c$$

now, substituting in, and solving for c with our **initial condition** we have,

$$\begin{aligned} 19.6 &= (0.8)(2)(0)^{\frac{1}{2}} + c \\ \therefore \quad c &= 19.6 \end{aligned}$$

Now we can state the full expression for the population at some year t,

$$P(t) = 1.6t^{\frac{1}{2}} + 19.6 \ .$$

20.3

Q: Find $\int(x^3 - x + 1)$ dx and $\int 3x^2$ dx.
A: In the first case, we can write,

$$\begin{aligned} \int (x^3 - x + 1) \text{ dx} &= \int x^3 \text{ dx} - \int x \text{ dx} + \int 1 \text{ dx} \\ &= \left(\frac{x^4}{4} + c_1\right) - \left(\frac{x^2}{2} + c_2\right) + (x + c_3) \\ &= \frac{x^4}{4} - \frac{x^2}{2} + x + c \end{aligned}$$

In the second case, we just have,

$$\begin{aligned} \int 3x^2 \text{ dx} &= 3\int x^2 \text{ dx} = 3\left(\frac{x^3}{3} + c_1\right) \\ &= 3\frac{x^3}{3} + c \\ &= x^3 + c \end{aligned}$$

(... notice the combining of constants.)

20.4

Q: *All the rules at once* Find

$$\int \left(5e^x - x^{-2} + \frac{3}{x}\right) \text{ dx} \quad (x \neq 0).$$

A: Using many of our rules covered so far ...

$$\begin{aligned} \int \left(5e^x - x^{-2} + \frac{3}{x}\right) \text{ dx} &= 5\int e^x \text{ dx} - \int x^{-2} \text{ dx} + 3\int \frac{1}{x} \text{ dx} \\ &= (5e^x + c_1) - \left(\frac{x^{-1}}{-1} + c_2\right) + (3\ln|x| + c_3) \\ &= 5e^x + \frac{1}{x} + 3\ln|x| + c \, . \end{aligned}$$

20.5

Q: *Substitution rule* Find the integral of the function $f(x) = 2x(x^2 + 1)$.
A: We can multiply out the integrand and solve:

$$\int 2x(x^2 + 1) \text{ dx} = \int (2x^3 + 2x) \text{ dx} = \frac{x^4}{2} + x^2 + c$$

Or, we notice that we can make the **substitution**, $u = x^2 + 1$, and notice that $\frac{du}{dx} = 2x$, which implies, $dx = \frac{du}{2x}$. So substituting and solving,

$$\begin{aligned} \int 2xu\frac{du}{2x} &= \int u \text{ du} = \frac{u^2}{2} + c \\ \therefore \quad \int 2x(x^2 + 1) \text{ dx} &= \frac{1}{2}(x^2 + 1)^2 + c \\ &= \frac{1}{2}x^4 + x^2 + c \, .\text{(1 incorporated into } c\text{)} \end{aligned}$$

20.6

Q: *The* really *useful substitution rule!* Find $\int 6x^2(x^3 + 2)^{99}$ dx.
A: ... not so easy to do by hand. However, suppose $u = x^3 + 2$; we have $\frac{du}{dx} = 3x^2$, so

$$\begin{aligned} \int 6x^2(x^3 + 2)^{99} \text{ dx} &= \int \left(2\frac{du}{dx}u^{99}\right) \text{ dx} = \int 2u^{99} \text{ du} \\ &= \frac{2}{100}u^{100} + c = \frac{1}{50}(x^3 + 2)^{100} + c \, . \end{aligned}$$

Fast!

20.7

Q: Evaluate $\int_1^5 3x^2\ \mathrm{dx}$.
A: We know that the **indefinite integral** is x^3+c, so we have the **definite integral** as,

$$\int_1^5 3x^2\ \mathrm{dx} = x^3\Big]_1^5 = (5)^3-(1)^3 = 125-1 = 124$$

20.8

Q: Suppose $f(x)=k(1-e^x)$, find $\int_a^b f(x)\ \mathrm{dx}$ (k is a constant).
A: We solve as normal, but being careful of the constant,

$$\begin{aligned}\int_a^b k(1-e^x)\ \mathrm{dx} &= k\left(x-e^x\right)\Big]_a^b \\ &= k(b-e^b)-k(a-e^a) \\ &= k(e^a-e^b+b-a)\end{aligned}$$

20.9

Q: Suppose $f(x)=e^{x/3}-3$, evaluate $\int_{-3}^6 f(x)\ \mathrm{dx}$ and compare it to the region that lies between the x-axis and $f(x)$ on the interval $[-3,6]$.
A: In the first case, we can ignore any crossings of the x-axis by $f(x)$, so,

$$\int_{-3}^6 e^{x/3}-3\ \mathrm{dx} = 3e^{x/3}-3x\Big]_{-3}^6 = (3e^2-18)-(3e^{-1}+9) = -5.936.$$

Next, we need to be careful – the integrand is negative on the interval $[-3,3\ln 3]$. Hence, we will chop up our region into two parts,

$$\begin{aligned}A &= -\int_{-3}^{3\ln 3} e^{x/3}-3\ \mathrm{dx} + \int_{3\ln 3}^{6} e^{x/3}-3\ \mathrm{dx} \\ &= -[(3e^{\ln 3}-9\ln 3)-(3e^{-1}+9)]+[(3e^2-18)-(3e^{\ln 3}-9\ln 3)] \\ &= 10.991+5.055 = 16.046\end{aligned}$$

... They are different! (why?)

Lecture 21

21.1

Q: Consider the equations,

$$(a)\ \left(\frac{dy}{dx}\right)^3+4y=12 \quad (b)\ \frac{d^2y}{dt^2}-y=0\ ,$$

what **order** and **degree** are these differential equations?
A:

a. we have $\frac{dy}{dx}$ which is the highest order derivative, so we have first-order, but it is raised to the third power, so we have third degree. ('A first-order differential equation of degree 3.')

b. In the second case we have a 'second-order differential equation of degree 1,' with the same reasoning.

21.2

Q: Find the general soluton to,

$$\frac{dy}{dx}=xy \quad \text{where} \quad y>0\,.$$

A: Rearranging, we have,

$$\int \frac{1}{y}\ \mathrm{dy} = \int x\ \mathrm{dx}$$

so solving,

$$\begin{aligned}\ln|y|+c_1 &= \frac{x^2}{2}+c_2 \\ \ln y &= \frac{x^2+c}{2} \\ y &= e^{\frac{x^2+c}{2}}\,.\end{aligned}$$

21.3

Q: Solve the equation, $\frac{dy}{dt}+2y=6$ by using separation of variables.
A: Proceed as before to separate,

$$\begin{aligned}\frac{dy}{dt} &= 6-2y = -2(y-3) \\ \therefore \quad \int \frac{1}{y-3}\ \mathrm{dy} &= \int -2\ \mathrm{dt}\end{aligned}$$

Then integrate and rearrange,

$$\begin{aligned}\ln|y-3|+c_1 &= -2t+c_2 \\ \therefore \quad e^{-2t+c} &= y-3 \\ \therefore \quad y &= e^{-2t}.e^c+3\end{aligned}$$

Letting e^c be the constant A, we have,

$$y=Ae^{-2t}+3$$

21.4

Q: Given the general solution, $y=Ae^{-2t}+3$, and initial condition $y(0)=10$, find the definite solution.
A: Substituting we have,

$$\begin{aligned}10 &= Ae^{(-2)(0)}+3 = A+3 \\ \therefore \quad y &= 7e^{-2t}+3\,.\end{aligned}$$

21.5

Q: Show that the general solution to the non-homogeneous first-order differential equation, $\frac{dy}{dt}+ay=b$ $(y>b/a)$ is

$y(t) = Ae^{-at} + \frac{b}{a}$. (Harder)
A:

$$\begin{aligned}
\frac{dy}{dt} &= b - ay = -a(y - \frac{b}{a}) \\
\int \frac{1}{y - \frac{b}{a}}\ \mathrm{dy} &= -\int a\ \mathrm{dt} \\
\ln|y - \frac{b}{a}| + c_1 &= -at + c_2 \\
e^{-at+c} &= y - \frac{b}{a} \\
y &= Ae^{-at} + \frac{b}{a}.
\end{aligned}$$

21.6

Q: Using the result of the previous example and assuming that $y(0) = y_0$, show that the definite solution is $y(t) = [y_0 - \frac{b}{a}]e^{-at} + \frac{b}{a}$.
A: We have,

$$y = Ae^{-at} + \frac{b}{a} ,$$

So, by substitution,

$$\begin{aligned}
y_0 = Ae^{-a(0)} + \frac{b}{a} &= A + \frac{b}{a} \\
\therefore \quad A &= y_0 - \frac{b}{a}
\end{aligned}$$

which gives us,

$$y(t) = \left[y_0 - \frac{b}{a}\right] e^{-at} + \frac{b}{a} .$$

21.7

Q: *Check* Solve (again) the differential equation $\frac{dy}{dt} + 2y = 6$ where $y(0) = 10$.
A: Using our new equation (**??**) for non-homogeneous first-order differential equations,

$$y(t) = \left[y_0 - \frac{b}{a}\right] e^{-at} + \frac{b}{a} .$$

we have,

$$\begin{aligned}
y(t) &= \left[10 - \frac{6}{2}\right] e^{-2t} + \frac{6}{2} \\
&= 7e^{-2t} + 3 .
\end{aligned}$$

... easy!

21.8

Q: The CIA fact-book[1] predicted Australia's population growth rate in 2006 to be 0.85%. Find a general equation for Australia's population at any given time.
A: Let $p(t)$ be Aust. pop at time t, so the rate of change of population (the growth rate) would be,

$$\frac{dp}{dt} = 0.0085p(t)$$

which is a differential equation of **exponential growth**, so we have,

$$p(t) = p_0 e^{0.0085t} .$$

21.9

Q: Using the equation derived from the previous example, find how long it would take at this rate for Australia's population to double.
A: So we have $p_1 = 2p_0$, that is, population at time 1 is twice that of time 0, or,

$$\begin{aligned}
2p_0 &= p_0 e^{0.0085t} \\
\therefore \quad \ln 2 &= 0.0085t \\
\therefore \quad t_{dbl} &= \frac{\ln 2}{0.0085} = 81.5 \text{ years}
\end{aligned}$$

21.10

Q: However, the fact-book also says that Australia's net migration is 3.85 migrants per 1000 population per year (2006). Given this fact, re-calculate the years to double the population of Australia.
A: So the actual growth rate of Australia's population is at $0.85 + 0.385 = 1.235\%$, so the total population equation is therefore,

$$p(t) = p_0 e^{0.01235t}$$

Giving an adjusted doubling time of,

$$t_{dbl} = \frac{\ln 2}{0.01235} = 56.1 \text{ years} .$$

Lecture 22

22.1

Q: Show that the limit of $N(t) = \left[N_0 - \frac{b}{a}\right] e^{\alpha t} + \frac{b}{a}$ is $\frac{b}{a}$ for $\alpha < 0$ and ∞ for $\alpha > 0$.
A: In the first case (exponential decay),

$$\lim_{t\to\infty} e^{-\alpha t} = \lim_{t\to\infty} \frac{1}{e^{\alpha t}} = 0$$

so we have

$$\lim_{t\to\infty} N(t) = \left[N_0 - \frac{b}{a}\right](0) + \frac{b}{a} = \frac{b}{a}$$

But in the second we have

$$\lim_{t\to\infty} e^{\alpha t} \to \infty \quad (!)$$

... so $\lim_{t\to\infty} y(t)$ is undefined, but goes to ∞.

22.2

[1] URL: `http://www.cia.gov/publications/factbook/index.html`

Q: By the method of separation of variables, prove the previous definition for limited growth.
A:

$$\begin{aligned} \frac{dN}{dt} &= k(M - N(t)) \\ \therefore \int \frac{1}{M-N}\,\mathrm{dN} &= \int k\ \mathrm{dt} \\ \therefore \quad -\ln|M-N| + c_1 &= kt + c_2 \\ \therefore \quad M - N &= e^{-kt+c} \\ \therefore \quad N(t) &= M - Ae^{-kt} \end{aligned}$$

22.3

Q: Suppose that a government economist works out that Australia's carrying capacity is 52 million people. Using the final constant of growth (internal and migration) from the last lecture, how long will it take for Australia's population to be at 90% of carrying capacity? (Assume current population is 20 million.)
A: From last lecture, we had a final r of 0.01235. Now, using a **limited exponential growth** equation of the form,

$$\frac{dp}{dt} = r(K - p) ,$$

where r is the constant of growth, and K is the carrying capacity; Definition **??** gives us,

$$\begin{aligned} p(t) &= K - (K - p_0)e^{-rt} \\ \therefore \quad t &= -\frac{1}{r}\ln\left(\frac{K-p}{K-p_0}\right) = -\frac{1}{0.01235}\ln\left(\frac{52-(0.9)(52)}{52-20}\right) \\ &= 147 \text{ years} \end{aligned}$$

22.4

Q: Show that for the logistic growth function,

$$p(t) = \frac{M}{1 + Ae^{-kt}} ,$$

if $p(0) = p_0$, then $A = \frac{M}{p_0} - 1$.
A: Setting $t = 0$ and equating to p_0 gives,

$$\begin{aligned} p_0 &= \frac{M}{1 + Ae^{-r(0)}} = \frac{M}{1 + A(1)} \\ \therefore \quad 1 + A &= \frac{M}{p_0} \\ \therefore \quad A &= \frac{M}{p_0} - 1 . \end{aligned}$$

22.5

Q: With our model fitted above, find the rate of population growth in the year 1788.
A: Recall our (adjusted) population **growth** equation,

$$\frac{dp}{dt} = kp\left(\frac{M - p(t - t_0)}{M}\right)$$

We have $M = 30$mil, $k = 0.03$, and $t_0 = 1630$. First, we need to find the (adjusted) population in 1788 by our estimated model:

$$p(1788) = \frac{30 \times 10^6}{1 + (30 \times 10^6/859 - 1)e^{-0.03(1788-1630)}} = 98,026$$

so by substitution we obtain,

$$\left.\frac{dp}{dt}\right|_{t=1788} = (0.03)(98,026)\left(\frac{30 \times 10^6 - 98,026}{30 \times 10^6}\right) = 2,931 \text{ people per year}$$

Lecture 23

23.1

Q: Given $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, find f_{x_1} and f_{x_2}.
A: To find f_{x_1}, we remember to **treat** x_2 **as constant**,

$$\frac{\delta y}{\delta x_1} = f_{x_1} = 6x_1 + x_2$$

and then to find f_{x_2} we remember to **treat** x_1 **as constant**,

$$\frac{\delta y}{\delta x_2} = f_{x_2} = x_1 + 8x_2$$

So the partial derivatives of f are $f_{x_1} = 6x_1 + x_2$ and $f_{x_2} = x_1 + 8x_2$.

23.2

Q: Find f_x and f_y given that $f(x, y) = (2x + 4)(y - 3)$.
A: Using the product rule, treating y as a constant, we have:

$$\begin{aligned} f_x &= (2x+4)(0) + (2)(y-3) \\ &= 2(y-3) \end{aligned}$$

And for f_y we have,

$$\begin{aligned} f_y &= (2x+4)(1) + (0)(y-3) \\ &= 2x + 4 \end{aligned}$$

23.3

Q: In the graphical example above, the function is $f(x, y) = 2 - (1 - x)^2 - (1 - y)^2$. Show that $f_{xy} = f_{yx} = 0$. That is, that the partial derivatives f_x and f_y do not change as the 'other' variable is changed.
A: We have,

$$\begin{aligned} f_x &= (-1)(2)(-1)(1-x) = 2(1-x) , \\ f_y &= (-1)(2)(-1)(1-y) = 2(1-y) , \end{aligned}$$

Taking the cross-partial derivatives,

$$\begin{aligned} \frac{\delta f_x}{\delta y} &= 0 \\ \frac{\delta f_y}{\delta x} &= 0 \end{aligned}$$

... as required.

23.4

Q: Find the second-order partial derivatives of the function $z = f(x, y) = x^3e^{-2y}$.
A: The first-order partial derivatives,

$$\begin{aligned} f_x &= 3x^2e^{-2y} \\ f_y &= -2x^3e^{-2y} \end{aligned}$$

And the second-order derivatives:

$$\begin{aligned} f_{xx} &= 6xe^{-2y} \\ f_{yy} &= (-2)(-2x^3)e^{-2y} = 4x^3e^{-2y} \\ f_{xy} &= (-2)(3x^2)e^{-2y} = -6x^2e^{-2y} \\ f_{yx} &= (3)(-2x^2)e^{-2y} = -6x^2e^{-2y} \end{aligned}$$

Note: $f_{xy} = f_{yx}$ (again!).

23.5

Q: *Chain rule* Find $\frac{\delta z}{\delta t}$ given that $z(x, y) = \frac{x+y}{2x^2}$ and $x(t) = 2t^2 + 1$ and $y(t) = 3t$.
A: So we have,

$$\frac{\delta z}{\delta t} = f_x \frac{\delta x}{\delta t} + f_y \frac{\delta y}{\delta t}$$

the partial derivatives are:

$$f_x = \frac{-(x + 2y)}{2x^3} \qquad f_y = \frac{1}{2x^2}$$

and the secondary functions give,

$$\frac{\delta x}{\delta t} = 4t \qquad \frac{\delta y}{\delta t} = 3$$

so we have,

$$\frac{\delta z}{\delta t} = \frac{-(x + 2y)}{2x^3}(4t) + \frac{1}{2x^2}(3)$$

23.6

Q: *Total differentiation* Find the total differential of $z = 3x^2 + xy - 2y^3$.
A: First we find the partial derivatives of z:

$$\begin{aligned} f_x &= 6x + y \\ f_y &= x - 6y^2 \end{aligned}$$

then we can write down (by the chain rule) the total differential:

$$\mathrm{dz} = (6x + y)\ \mathrm{dx} + (x - 6y^2)\ \mathrm{dy}$$

Lecture 24

24.1

Q: Find the extreme values of the function,

$$z = f(x_1, x_2) = 2x_1^2 + x_1x_2 + 4x_2^2 \ .$$

A: First, write down the partial derivatives:

$$\begin{aligned} f_{x_1} &= 4x_1 + x_2 \\ f_{x_2} &= x_1 + 8x_2 \end{aligned}$$

Solve for any (x_1, x_2) that satisfy $f_{x_1} = f_{x_2} = 0$,

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 1 & 8 & 0 \end{array}\right], \ |\mathbf{A}| = 31,$$

So we have (trivially), $x_1^* = x_2^* = 0$.

24.2

Q: For the plotted function used previously,

$$z = f(x, y) = 2 - (x - 1)^2 - (y - 1)^2 \ ,$$

find the location of the local maxima.
A: As before,

$$\begin{aligned} f_x &= -2(x - 1) \\ f_y &= -2(y - 1) \end{aligned}$$

Solve for any (x, y) that satisfy $f_x = f_y = 0$,

$$\left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 2 & 2 \end{array}\right], \ |\mathbf{A}| = 4,$$

By the adjoint method, we have:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So we have, $x^* = y^* = 1$.

24.3

Q: Find the extreme value(s) of the function $z = f(x, y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$.
A: So we will need: $f_x, f_y, f_{xx}, f_{yy}, f_{xy}$:

$$\begin{aligned} f_x &= 24x^2 + 2y - 6x & f_y &= 2x + 2y \\ f_{xx} &= 48x - 6 & f_{yy} &= 2, \quad f_{xy} = 2 \end{aligned}$$

First order necessary conditions, we solve for (x_0, y_0):

$$\begin{aligned} f_y = 0 = 2x + 2y \quad &\Rightarrow \quad y = -x \\ \therefore \quad 24x^2 - 8x = 0 \quad &\Rightarrow \quad x^* = 0 \text{ or } \frac{1}{3} \\ \therefore \quad \text{Options} \quad \ldots \quad & (0, 0) \text{ and } (\frac{1}{3}, -\frac{1}{3}) \end{aligned}$$

Second-order sufficient conditions: for (0,0) $f_{xx} = -6$, but $f_{yy} = 2$, implying: $f_{xx}f_{yy} - f_{xy}^2 = -16$! However, for $(\frac{1}{3}, -\frac{1}{3})$, $f_{xx} = 10$ and $f_{yy} = f_{xy} = 2$, so $f_{xx}f_{yy} - f_{xy}^2 = 16$ hence it satisfies a **minimum** at the point $(\frac{1}{3}, -\frac{1}{3}, \frac{23}{27})$.

24.4

Q: Using the function

$$z = f(x, y) = 2 - (x - 1)^2 - (y - 1)^2 \ ,$$

and the constraint $2 = y + \frac{2}{5}x$, find the new optimum point.
A: We can proceed by substitution, knowing that $y = 2 - \frac{2}{5}x$, so:

$$\begin{aligned} z &= 2 - (x - 1)^2 - [(2 - \frac{2}{5}x) - 1]^2 \\ \therefore \quad \frac{\mathrm{dz}}{\mathrm{dx}} &= -(2)(x - 1) - (\frac{2}{5})(2)(1 - \frac{2}{5}x) \\ &= 2\frac{4}{5} - 2\frac{8}{25}x \end{aligned}$$

Solve (first order condition) by setting $\frac{\mathrm{dz}}{\mathrm{dx}} = 0$, gives $x^* = 1\frac{6}{29} \approx 1.21$, so $y^* = 1\frac{15}{29} \approx 1.52$ and $z^* = 1.69$.

24.5

Q: Check the previous example, using the Lagrange Multiplier method.
A: Step 1: With $g(x, y) = y + \frac{2}{5}x$ and $c = 2$, set up the Lagrangian,

$$L = 2 - (x-1)^2 - (y-1)^2 + \lambda[2 - (y + \frac{2}{5}x)]\,,$$

Step 2: find the first-order conditions:

$$f_x = -2(x-1) \quad f_y = -2(x-1) \quad g_x = \frac{2}{5} \quad g_y = 1$$

$$\begin{aligned} L_x &= & f_x - \lambda g_x & \quad = -2(x-1) - \lambda\frac{2}{5} = 0 \\ L_y &= & f_y - \lambda g_y & \quad = -2(y-1) - \lambda = 0 \\ L_\lambda &= & [c - g(x,y)] & \quad = 2 - (y + \frac{2}{5}x) = 0 \end{aligned}$$

Step 3: solve (using matrix algebra):

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{2}{5} & 2 \\ 0 & 2 & 1 & 2 \\ \frac{2}{5} & 1 & 0 & 2 \end{array}\right] \quad \text{giving :} \quad \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 1.21 \\ 1.52 \\ -1.03 \end{bmatrix}$$

24.6

Q: Check the previous example for second-order conditions.
A: Write down the partial differentials we need:

$$\begin{array}{cc} g_x = \frac{2}{5} & g_y = 1 \\ L_x = -2(x-1) - \lambda\frac{2}{5} & L_{xx} = -2 \\ L_y = -2(y-1) + \lambda & L_{yy} = -2 \\ L_{xy} = 0 & L_{yx} = 0 \end{array}$$

Construct $\bar{H}$ and find the determinant:

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & \frac{2}{5} & 1 \\ \frac{2}{5} & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 2.32 \qquad \text{(check!)}$$

Since $|\bar{H}|$ is positive everywhere, we conclude that the point satisfying necessary conditions found before is a maxima.