The acyclic edge chromatic number of a random d-regular graph is $d+1$

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AMS classification: 05C15 (primary: graph colouring) 68R05 (secondary: combinatorics)
Keywords: colouring, acyclic colouring, edge colouring, regular graphs, random graphs

Abstract

We prove the theorem from the title: the acyclic edge chromatic number of a random d-regular graph is asymptotically almost surely equal to $d+1$. This improves a result of Alon, Sudakov and Zaks and presents further support for a conjecture that $\Delta(G)+2$ is the bound for the acyclic edge chromatic number of any graph G. It also represents an analogue of a result of Robinson and the second author on edge chromatic number.

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[*] Partially supported by the Project LN00A056 of the Czech Ministery of Education and performed in part while visiting CRM, Bellatera, Spain.

[†] Research supported in part by the Australian Research Council and performed in part while visiting CRM, Bellatera, Spain.

1 Introduction

The acyclic chromatic number $A(G)$ of a graph G is the minimum number colours which suffice for a proper colouring of vertices of G in such a way that no cycle in G is coloured by two colours only. (The acyclic chromatic number is also denoted by $\chi_a(G)$, we adopt the notation of [2]). The acyclic chromatic number was defined and studied in the context of planar graphs by Grünbaum and this led to extensive research, see e.g. [4, 9, 1, 11]. It is known that $A(G) \leq 5$ for every planar graph G [4] and this has been used to get the best known bounds for other graph parameters such as oriented chromatic number and star chromatic number. It is also known, and easy to see, that $A(G)$ cannot be bounded in terms of chromatic number $\chi(G)$ (consider the bipartite graph which we get by subdividing each edge of a complete graph by a single vertex). However $A(G)$ can be bounded by a function of maximal $\chi(H)$ where H is a minor of G [10]. This further supports a remarkable insight of Grünbaum who initiated the study of $A(G)$ in a geometric context.

The papers [2, 3] study acyclic chromatic number for random graphs as well as the edge analogue $A'(G)$ of $A(G)$. Let us introduce it in the following form: given a graph G we denote by $A'(G)$ the minimum number of matchings which suffice to cover all edges of G in such a way that the union of any two matchings does not contain any circuit. $A'(G)$ is called *acyclic edge chromatic number* of G.

In contrast to $A(G)$, the acyclic edge chromatic number $A'(G)$ behaves very much like the edge chromatic number $\chi'(G)$. Clearly

$$\Delta(G) \leq \chi'(G) \leq A'(G)$$

for any graph G and it has been conjectured in [2] that $A'(G) \leq \Delta(G) + 2$ for any graph G. (Recall that we have $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ by Vizing's theorem, [7].) A property of a random d-regular graph is true *asymptotically almost surely* if its probability tends to 1 as the number of vertices tends to infinity.

We prove the following.

Theorem 1 *For any fixed $d \geq 2$ the acyclic edge chromatic number of a random d-regular graph is asymptotically almost surely equal to $d + 1$.*

This is a strengthening of [3] where this result is proved for graphs with an even number of vertices (while for odd number of vertices [3] gives $d+2$ only). Theorem 1 also provides the acyclic edge colouring equivalent of the result proved in [12]: $\chi'(G)$ is for d-regular graphs a.a.s. equal to d (if the number of vertices is even). The proof relies on the contiguity of a matching model [12] and [6] and of a Hamilton decomposition model [8] with the uniform model for random d-regular graphs. (See

[13] for an introduction to models of random regular graphs and their contiguity.) These contiguity results allow us to treat the cases of even and odd n in a similar way to achieve the same bound, unlike the approach in [3].

Let us remark that the acyclic edge chromatic number $A'(G)$ is at least $d+1$ for any d-regular graph G with $d \geq 2$: if G has odd number of vertices then $A'(G) \geq \chi'(G) \geq d+1$, whilst if G has even number of vertices and $\chi'(G) = d$ then any two colours in a proper edge d-colouring of G induce a 2-factor, which of course contains a cycle. Thus also in this case $A'(G) > d$.

It is conjectured in [3] that every graph G with maximum degree $\Delta(G)$ satisfies $A'(G) \leq \Delta(G) + 2$. This is complemented in [3] by a question that perhaps for a d-regular graph it is even true that $A'(G) = d+1$ with the unique exception of K_n, n even. Both of these conjectures are supported by the result in this paper.

2 Proof of Theorem 1

As explained above it suffices to prove the upper bound. First suppose $n = |V(G)|$ is even. (Although this was proved in [3], we prove it again here using a different method in order to show a less complicated version of our proof for n odd.) Then G is contiguous to a superposition of d independent uniformly random perfect matchings $M_1, M_2, \ldots, M_d$ of the n vertices, restricted to no double edges forming (which is an event with asymptotically constant probability, see for example [13]). So we only have to prove that the statement holds for the graph with edge set $M_1 \cup \cdots \cup M_d$, conditional on $M_i \cap M_j = \emptyset$ for every $1 \leq j < i \leq d$.

Begin with each matching M_i having a separate colour i. Consider adding the matchings one at a time: fixing $M_1, \ldots, M_{i-1}$, take M_i at random. We prove by induction on i that only $O(\log n)$ edges of colour $d+1$ are required, in a colouring of $M_1 \cup \cdots \cup M_i$with no bichromatic cycles. Note that a.a.s. $O(\log n)$ cycles are formed by M_i with any of the previous matchings M_j. (This is well known, and follows from the corresponding property of random permutations, for which see for example Kim and Wormald [8, Lemma 4.1]). Break each such cycle C by choosing randomly one of the edges of $C \cap M_i$, to colour $d+1$. (Two cycles are permitted to choose the same edge for this purpose, indeed that will sometimes happen, and still suffices to break the cycles formed by M_i with previous matchings.) Since M_i occurs uniformly at random, all vertices are equally likely to be incident with these "new" edges of colour $d+1$. This is true for each M_j. So the probability for each vertex is $O(\log n/n)$, and hence a.a.s. all these $O(\log n)$ new edges of colour $d+1$ have distance at least 3 (distance considered in the line graph of the graph formed by the first $i-1$ matchings) from all of the "old" ones of colour $d+1$ (of which there are $O(\log n)$, by induction). It follows that a.a.s. there is no bichromatic cycle formed which uses both new and old edges of colour $d+1$, together with some other colour.

On the other hand, a cycle with colours $d+1$ and i would require some edge of M_i to be incident with at least two of the old $O(\log n)$ edges of colour $d+1$, which is an event with probability $O(\log^2 n/n)$.

The only remaining possibility for a bichromatic cycle is to use one of the colours from 1 to $i-1$ together with the new edges of colour $d+1$. To deal with this takes a little more effort. Consider the probability p that a pair of vertices are joined by two distinct paths of length less than $t = \log\log n$ in the graph formed by $M_1 \cup \cdots \cup M_{i-1}$. Since the matchings are random, and the number of vertices reachable from one vertex by such paths is $O(d^t) = (\log n)^{O(1)}$, we have $p = (\log n)^{O(1)}/n^2$. So a.a.s. no edge of M_i joins such a pair of vertices, and thus we may assume that no edge of M_i is in two distinct cycles of $M_1 \cup \cdots \cup M_i$ of length less than t.

Now consider some $j < i$ and a cycle C of length $2m$ in $M_i \cup M_j$. First consider the case that $2m < t$. Any other cycle C' of $M_i \cup M_{j'}$ sharing an edge with C will have, by the assumption above, length at least t, so the probability that the shared edge is newly coloured $d+1$ is at most $1/t$. Summing over j' shows that the probability that all edges of $C \cap M_i$ are newly coloured $d+1$ is $O(t^{-m+1})$. (The extra term $+1$ in the exponent is for the edge which was newly coloured $d+1$ to break C.) On the other hand, the expected number of cycles C of length at most t formed by two random matchings — in this case M_i and M_j — is $O(\log t)$. Thus, a.a.s. there is no such C which becomes bichromatic.

Secondly, consider the case $2m > t$. The expected number of cycles of length at most 160 in $M_i \cup M_{j'}$ is bounded, and so a.a.s. there are at most $t/4$ of them, summed over all j'. Thus, we may assume that at least $t/4+1$ edges of M_i in C are not in cycles of any $M_i \cup M_{j'}$ of length at most 160. Hence the probability that all edges of C are newly coloured $d+1$ is at most $81^{-t/4} = 3^{-t}$. Multiplying by the number of such C, which as explained above is $O(\log n) = O(e^t)$, shows that the expected number of C giving problems is $o(1)$.

Thus a.a.s. no cycle has only two colours, and the required colouring with $d+1$ colours goes through by induction on i.

Now consider $n = |V(G)|$ odd. Then d is even, and (by the main result in [8]) G is contiguous to a superposition of $d/2$ random Hamilton cycles. Colour each by giving one colour to each of two near-perfect matchings of $n-1$ vertices, alternating around the cycle, and giving the last colour, $d+1$, to the remaining edge (which we can assume is chosen randomly). Then proceed in the complete analogy to the case of n even. Assume the first $i-1$ Hamilton cycles have been selected, and the bichromatic cycles broken, and choose the i-th Hamilton cycle H_i at random. Although its two near-perfect matchings, M_{2i-1} and M_{2i}, of colours $2i-1$ and $2i$, are correlated with each other, each occurs uniformly at random with respect to matchings occurring in earlier Hamilton cycles. Two random near-perfect matchings will again form $O(\log n)$ cycles a.a.s. (Perhaps the easiest way to see this is to augment each matching by an edge from the unmatched vertex to a new vertex, thereby giving two random perfect matchings, and use the result mentioned above.) Hence each matching a.a.s. requires $O(\log n)$ edges to be recoloured $d+1$ to break all cycles with earlier matchings.

Part of the argument for n even goes through exactly the same: bichromatic cycles using both new and old edges of colour $d+1$ a.a.s. do not occur, and nor does a cycle of colours $d+1$ and $2i$ or $2i-1$. Here we need to observe also that the extra single edge of H_i not in $M_{2i-1} \cup M_{2i}$, of colour $d+1$, does not interefere. But given either M_{2i-1} or M_{2i}, this single edge is uniformly distributed on all remaining available spaces, so there is no problem. The only case which is different is that of new edges of colour $d+1$ in collusion with edges of one of the colours $j \le 2i-2$. As with the case of n odd, we may assume that no edge of H_i is in two distinct cycles of $H_1 \cup \cdots \cup H_i$ of length less than t. Consider some $j < i$ and a cycle C of length m in $H_i \cup M_j$ for some $j \le 2i-2$, and define $t = \log\log n$ as before.

The only part of this argument which essentially changes is in the case that $2m < t$. The expected number of cycles C of length at most t in $H_i \cup M_j$ is $o(2^t)$ (as there are $[n]_u/2u$ cycles of length u in K_n and the probability a given set of $u \le \log\log n$ edges is present in $H_i \cup M_j$ is easily shown to be $O((2/n)^u)$). Thus, a.a.s. there is no such C which becomes bichromatic.

Secondly, consider the case $2m > t$. Again, the proof is similar to the even case. The expected number of cycles of length at most 160 in $M_{i'} \cup M_{j'}$ (for $i' = 2i-1$ or $2i$, and $j' < 2i-1$) is bounded, and so a.a.s. there are at most $t/4$ of them, summed over all j'. Thus, we may assume that at least $t/4+1$ edges of $M_{i'}$ in C are not in cycles of $M_{i'} \cup M_{j'}$ of length at most 160, for any $j' < 2i-1$. Hence the probability that all edges of C are newly coloured $d+1$ is at most $81^{-t/4} = 3^{-t}$. Multiplying by the number of such C, which as in the odd n case is $O(\log n) = O(e^t)$, the expected number of C giving problems is $o(1)$.

Thus a.a.s. no cycle has only two colours, and the required colouring with $d+1$ colours goes through by induction on i. ∎

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