Volume Conjecture Seminar 1

An introduction to “An introduction to the volume conjecture”

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Abstract. To a knot, we can associate algebraic invariants such as knot polynomials or geometric invariants such as the volume of the knot complement. It wasn’t until about ten years ago — when the volume conjecture was speculated — that there was a reason to believe that these two concepts were related. In this talk, I’ll give a basic introduction to knot theory, culminating in the statement of the volume conjecture.

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0 The big picture

A very useful survey article is *An introduction to the volume conjecture* by Hitoshi Murakami, which is available at arXiv:1002.0126v1 [math.GT]. Today, we won’t be careful in distinguishing between knots and links — when we say *knot*, we will sometimes allow it to have multiple components.

$$\begin{tikzcd}[column sep=small]
& \begin{array}{c}\text{KNOTS}\\ \{S^1 \hookrightarrow S^3\}/\text{ambient isotopy}\end{array} \arrow[dl, "\text{algebra}"'] \arrow[dr, "\text{geometry}"] & \\
\text{KNOT POLYNOMIALS} \arrow[rr, dashed, leftrightarrow, "\text{volume}", "\text{conjecture}"'] & & \text{HYPERBOLIC VOLUMES}
\end{tikzcd}$$

1 Knot projections

You can project a knot in three-dimensional space to the two-dimensional plane to generically obtain a 4-valent planar graph with under-crossings and over-crossings.

Example. Pictured below are knot projections of the unknot, the left-handed trefoil, and the figure-eight knot, respectively.

Question. When do two projections describe the same knot?

Two projections obviously describe the same knot if they are related by ambient isotopy of the plane or the following three moves, known as *Reidemeister moves*.

(I) twist/untwist (II) poke/unpoke (III) slide

Theorem (Reidemeister, 1927)**.** *Two knot projections describe equivalent knots if and only if they are related by ambient isotopy of the plane and Reidemeister moves.*

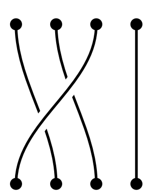
Proof. Suppose that you have two knot projections which describe equivalent knots.

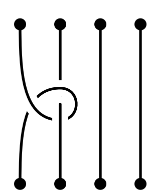
- Consider a "generic" ambient isotopy from one knot to the other.
- Watch the movie made by the projection.
- The knot projection changes combinatorially only when a region is created or destroyed.
- Such a region generically has 1, 2 or 3 sides — these correspond to the Reidemeister moves. □

2 Knots from braids

An *n-braid* consists of n strings running down the page, from n points ordered from left to right at the top to n points ordered from left to right at the bottom, up to ambient isotopy.

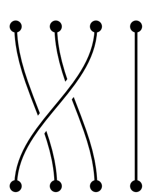
Example. The diagram below left is a simple example of a braid. The diagram below right is not an example of a braid, because the strings do not all run down the page (and cannot be made to do so after ambient isotopy).



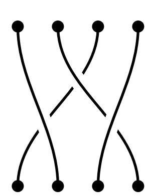


Fact. *The n-braids form an infinite group B_n.*

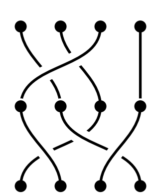
- Composition: The braid $\beta_1 \cdot \beta_2$ is obtained by putting the braid β_1 above the braid β_2.



β_1

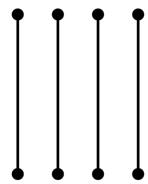


β_2

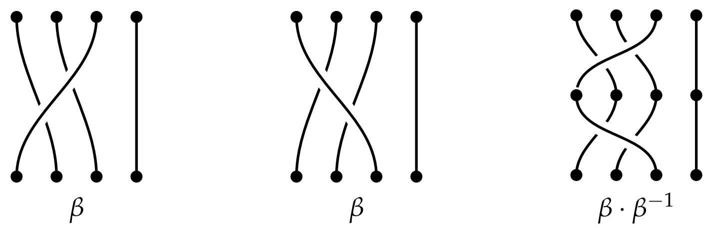


$\beta_1 \cdot \beta_2$

- Identity: The identity braid consists of strings which run directly down the page.



- Inverse: The braid β^{-1} looks like the projection of the braid β reflected in a horizontal axis.



After "jiggling" the projection of a braid, the crossings occur at different heights. So an n-braid decomposes into basic twists σ_k and σ_k^{-1} for $k = 1, 2, \ldots, n-1$.

It's easy to check that these twists must satisfy the following *braid relations*.

- $\sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i$ for $|i-j| \geq 2$
- $\sigma_k \cdot \sigma_{k+1} \cdot \sigma_k = \sigma_{k+1} \cdot \sigma_k \cdot \sigma_{k+1}$ for $k = 1, 2, \ldots, n-2$

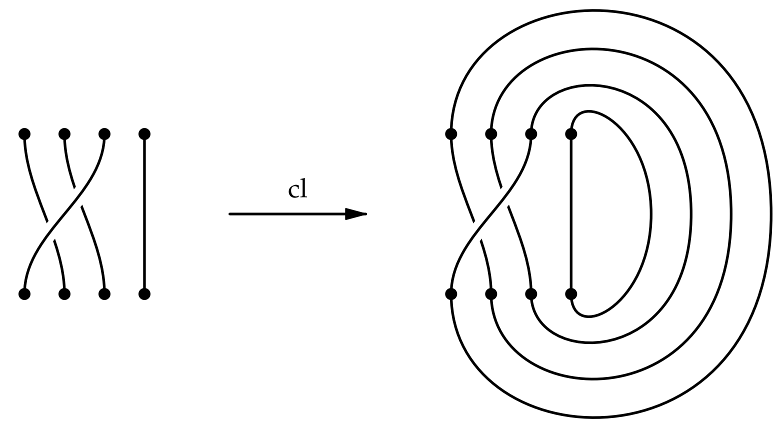
Theorem. *The braid group has the presentation* $B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \textit{braid relations} \rangle$.

Proof. As with knots and links, we only need to consider relations coming from ambient isotopy of the plane and Reidemeister moves.

- Ambient isotopy: Relations coming from ambient isotopy correspond to the braid relations of the first type.

- Reidemeister I: This move never occurs, since the strings of a braid always run down the page.
- Reidemeister II: This move corresponds to the trivial relation $\sigma \cdot \sigma^{-1} = \sigma^{-1} \cdot \sigma = \text{id}$.
- Reidemeister III: This move corresponds to the braid relations of the second type. □

There is a simple way to obtain a knot from a braid. To make a knot or link from a braid β, simply take its *closure*, which we'll denote by $\text{cl}(\beta)$.



Question. Which knots and links are closures of braids?

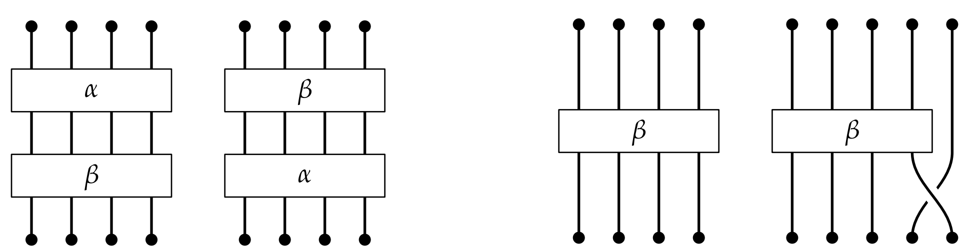
Theorem (Alexander, 1923)**.** *Every knot and link is the closure of a braid.*

The proof is not difficult and appears in Chapter 5 of *The Knot Book* by Colin C. Adams.

Question. When do two braids represent the same knot or link?

Two braids obviously represent the same knot or link if they are related by the following two moves.

- Conjugation: If $\alpha, \beta \in B_n$, then $\text{cl}(\alpha \cdot \beta) = \text{cl}(\beta \cdot \alpha)$.
- Stabilisation: If $\beta \in B_n$, then $\text{cl}(\beta) = \text{cl}(\beta \cdot \sigma_n^{\pm 1})$ where $\beta \cdot \sigma_n^{\pm 1} \in B_{n+1}$.



Theorem (Markov, 1936)**.** *Given two braids β_1 and β_2, the closures $cl(\beta_1)$ and $cl(\beta_2)$ represent the same knot or link if and only if β_1 and β_2 are related by conjugations, stabilisations and destabilisations.*

The proof is rather involved and appears in Chapter 2 of *Braids, Links, and Mapping Class Groups* by Joan S. Birman.

3 Knot polynomials

The idea is to associate a value to a knot projection which is invariant under Reidemeister moves. Another approach is to associate a value to a braid which is invariant under conjugations, stabilisations and destabilisations. This will produce a knot invariant $f(K)$ for which

- $f(K_1) \neq f(K_2) \Rightarrow K_1 \neq K_2$; and
- $f(K_1) = f(K_2) \Rightarrow K_1 = K_2$ or $K_1 \neq K_2$.

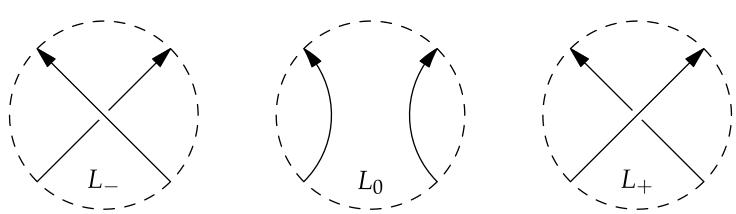
There are many knot invariants, but the most important ones are often polynomials.

- *Alexander polynomial* (1923)
  Let X be the infinite cyclic cover of the knot complement of K. The Alexander polynomial arises from considering the action of the covering transformation on the first homology $H_1(X; \mathbb{Z})$.
- *Jones polynomial* (1984)
  Vaughan Jones discovered the Jones polynomial while studying braid group representations and operator algebras. The Jones polynomial and its "coloured versions" are the knot invariants which we will be most concerned with.
- *HOMFLY polynomial* (1985)
  The HOMFLY polynomial can be defined using a simple combinatorial relation on the knot projection, known as a *skein relation*. The Alexander polynomial and the Jones polynomial can be obtained by appropriate substitutions from the HOMFLY polynomial. The HOMFLY polynomial is related to Chern–Simons gauge theories for $SU(N)$.
- *Kauffman polynomial* (1987)
  The Kauffman polynomial can also be obtained using a skein relation. The Jones polynomial can be obtained by appropriate substitution from the Kauffman polynomial. The Kauffman polynomial is related to Chern–Simons gauge theories for $SO(N)$.

4 The Jones polynomial

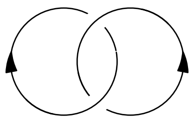
To an oriented link K, we associate a Laurent polynomial $J(K)$ in the variable $q^{1/2}$. It turns out that the Jones polynomial[1] $J(K)$ can be uniquely defined using the following local combinatorial rules on any given projection of K.

- Normalisation: $J(\text{unknot}) = 1$
- Skein relation: $(q^{1/2} - q^{-1/2})\, J(L_0) = q^{-1}\, J(L_+) - q\, J(L_-)$

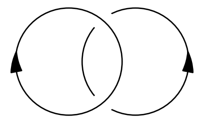


[1]When we want to make the dependence on q explicit, we will write $J(K; q)$.

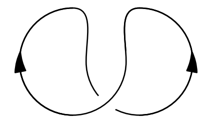
Example. Suppose that you want to calculate the Jones polynomial of the Hopf link. We can choose L_+ to be the Hopf link, L_- to be the unlink and L_0 to be the unknot, as shown in the diagram below.



L_+



L_-

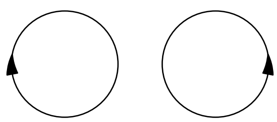


L_0

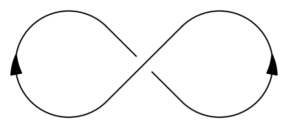
The skein relation then gives us the equation $(q^{1/2} - q^{-1/2})\, J(\text{unknot}) = q^{-1}\, J(\text{Hopf link}) - q\, J(\text{unlink})$ which simplifies to

$$J(\text{Hopf link}) = q^2\, J(\text{unlink}) + q^{3/2} - q^{1/2}.$$

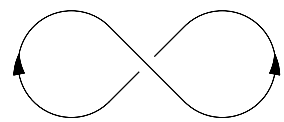
It remains to calculate the Jones polynomial of the unlink. We can choose L_0 to be the unlink, L_+ to be the unknot and L_- to be the unknot, as shown in the diagram below.



L_0



L_+



L_-

The skein relation then gives us the equation $(q^{1/2} - q^{-1/2})\, J(\text{unlink}) = q^{-1}\, J(\text{unknot}) - q\, J(\text{unknot})$ which simplifies to $J(\text{unlink}) = -q^{1/2} - q^{-1/2}$. Substituting this into the equation above, we find that

$$J(\text{Hopf link}) = -q^{1/2} - q^{5/2}.$$

There is a way to construct knot invariants from algebraic objects known as *quantum groups*, which can be thought of as deformations of Lie algebras. For a Lie algebra $\mathfrak{g}$, one can define a corresponding quantum group $U_q(\mathfrak{g})$, where q denotes a complex parameter which matches the q appearing in the Jones polynomial. One obtains a knot invariant for every representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ which uses the structure of the quantum group $U_q(\mathfrak{g})$. In particular, when $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, there exists a unique N-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, so we obtain a sequence of knot invariants $J_N(K)$. It turns out that the usual Jones polynomial corresponds to J_2 while in general, J_N is known as the *N-dimensional coloured Jones polynomial*. In future, we will see how to rigorously define the coloured Jones polynomials, since they are a crucial ingredient of the volume conjecture.

5 Geometry of the knot complement

Given a knot $K \subseteq S^3$, the *knot complement* $S^3 \setminus K$ is naturally a three-manifold. One of the main themes of modern three-manifold topology is *geometrisation*. In two dimensions, geometrisation takes the following form: every oriented closed surface possesses a unique complete metric with curvature -1, 0, or $+1$. These three cases correspond to hyperbolic, Euclidean, and spherical geometry, respectively.

The picture in three dimensions turns out to be far more complicated. It was conjectured by Thurston that every oriented prime closed three-manifold can be cut along tori[2] so that the interior of each

[2]Such a decomposition of a three-manifold is often called a *JSJ decomposition* after Jaco, Shalen and Johannson.

resulting piece has one of eight model geometries. We now know that Thurston's geometrisation conjecture is true, thanks to the work of Perelman. Fortunately, there is also a geometrisation theorem for knot complements due to Thurston.

Theorem (Thurston, late 1970s)**.** *Every knot is a torus knot, a satellite knot, or a hyperbolic knot.*

To understand this trichotomy, we need the following definitions.

- A *torus knot* is one which lies on the surface of an unknotted torus.
- A *satellite knot* is one which is obtained by taking a knot lying non-trivially inside a solid torus and then knotting the solid torus.
- A *hyperbolic knot* is one whose complement admits a complete hyperbolic metric.

It follows from the geometrisation theorem for knot complements that many knots are hyperbolic. In fact, there are only 32 knots which are not hyperbolic among the $1,701,936$ prime knots with at most sixteen crossings. Since the hyperbolic structure on a knot complement is guaranteed to be unique, any property extracted from it must be a knot invariant. In particular, the hyperbolic volume of a knot complement is a knot invariant. The software SnapPea is able to calculate hyperbolic volumes of knot complements to great precision.

6 The volume conjecture

The volume conjecture is a relatively recent development in mathematics — here is some of its history.

- In 1995, Kashaev discovered a sequence of knot invariants $f_N(K) \in \mathbb{C}$ which use the quantum dilogarithm function.
- In 1997, Kashaev observed that his knot invariants seem to have exponential growth with respect to N. More precisely, he conjectured that for hyperbolic knots,

$$2\pi \lim_{N\to\infty} \frac{\log |f_N(K)|}{N} = \text{hyperbolic volume of } S^3 \setminus K.$$

- In 2001, Murakami and Murakami proved that Kashaev's invariants are related to coloured Jones polynomials via the equation

$$f_N(K) = J_N(K)|_{q=\exp(2\pi i/N)}\,.$$

Putting all of this together, we obtain the following statement of the volume conjecture.

Conjecture (Volume conjecture)**.** *For a hyperbolic knot K,*

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K)|_{q=\exp(2\pi i/N)}}{N} = \textit{hyperbolic volume of } S^3 \setminus K.$$

There are now many formulations of the volume conjecture, many of which are generalisations of the original conjecture above. In particular, there is a compelling generalisation which applies to any knot, whether or not it is hyperbolic. One simply replaces the hyperbolic volume of the knot complement with the *simplicial volume*, otherwise known as the *Gromov norm*.

Some people in the business believe that the volume conjecture will not remain a conjecture for very long. However, it appears that the conjecture is just the shadow of a very deep connection between algebra and geometry.

The volume conjecture has been proved for

- the figure-eight knot by Ekholm;
- the 5_2 knot by Kashaev and Yokota;
- Whitehead doubles of torus knots by Zheng;
- torus knots by Kashaev and Tirkkonen;
- torus links of type $(2, 2m)$ by Hikami;
- knots and links with volume zero by van der Veen;
- the Borromean rings by Garoufalidis and Lê;
- twisted Whitehead links by Zheng;
- Whitehead chains by van der Veen; and
- the satellite link around the figure-eight knot with pattern the Whitehead link by Yamazaki and Yokota.

7 Where are we going?

In this seminar series, I hope that we'll be able to cover the following topics. All of them play a part in the ongoing mathematical development which is inspired by the volume conjecture.

- Hyperbolic geometry, knot complements, and how to calculate volumes of three-manifolds.
- The character variety of a knot complement, the A-polynomial, and the AJ conjecture.
- Quantum algebra, the Yang–Baxter equation, and the coloured Jones polynomial.
- The relation between Chern–Simons gauge theory and the Jones polynomial and between Chern–Simons invariants and hyperbolic volume.
- Eynard–Orantin invariants and their speculated relation to the AJ and volume conjectures.