VARIATIONS ON VERTICES AND VORTICES

Norm Do

The University of Melbourne

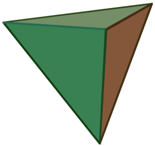
5 August 2011

Have you ever tried to count how many panels there are on a soccer ball? Have you ever wondered what the hairy ball theorem is and whether it applies to you? Have you ever thought about how curvaceous a person can be? If you've read this much of the abstract, then you should definitely come along to the seminar and learn about the amazing Euler characteristic!

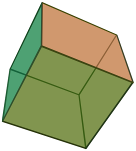
WHAT IS A POLYHEDRON?

Naively speaking, a polyhedron is a shape made from vertices, edges and faces.

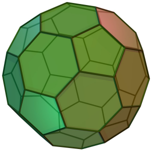
Some nice polyhedra



tetrahedron



cube



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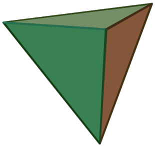

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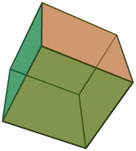

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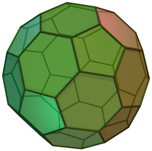
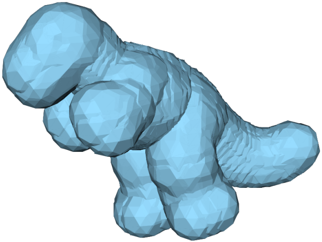


soccer ball

A more scary polyhedron



PROPERTIES OF POLYHEDRA

Let's count the number of vertices (V), edges (E) and faces (F) of these polyhedra.

polyhedron	V	E	F
tetrahredron	4	6	4
cube	8	12	6
soccer ball	60	90	32
dinosaur	?	?	?

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Euler's formula

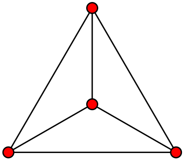
If a polyhedron has V vertices, E edges and F faces, then

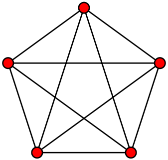
$$V - E + F = 2.$$

PLANAR GRAPHS

A graph consists of some vertices and some edges which join vertices in pairs.

A graph is called planar if you can draw it in the plane without any edges crossing.


planar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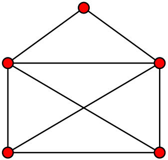
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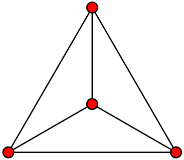

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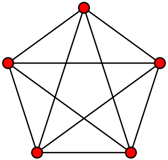

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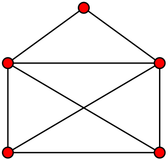

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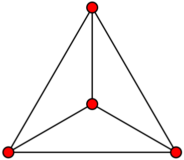

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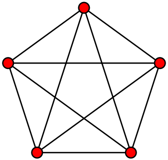

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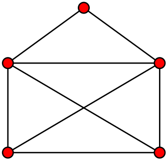

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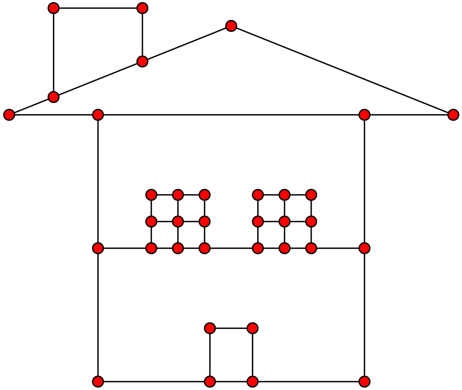
Fact

The vertices and edges of a polyhedron form a planar graph.

A SKETCH PROOF OF EULER'S FORMULA



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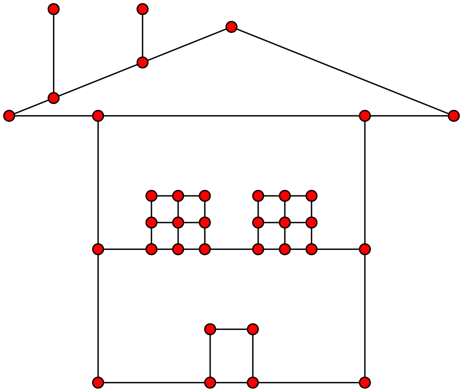

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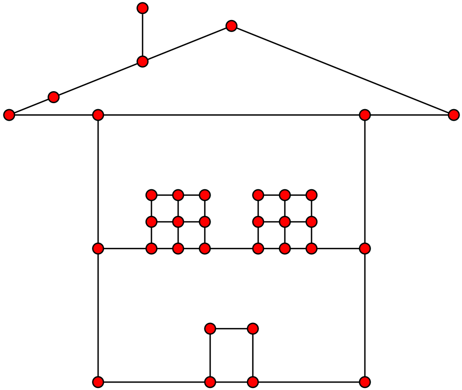

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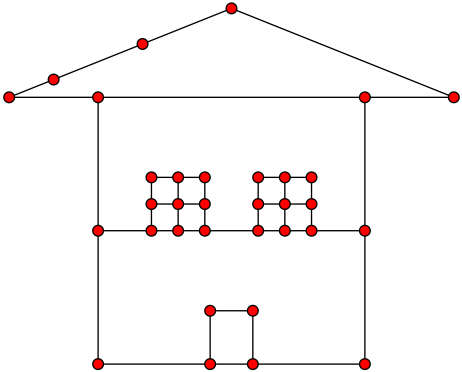

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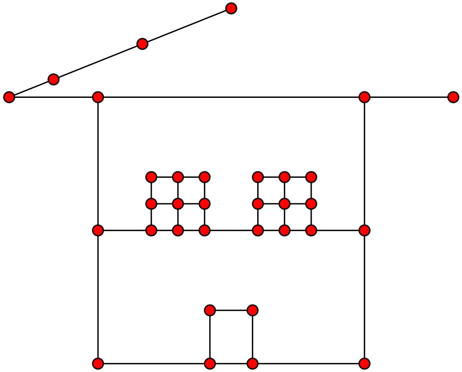

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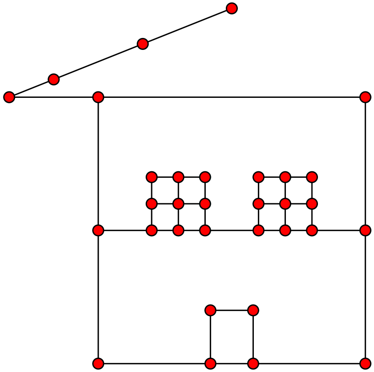

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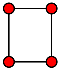

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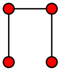

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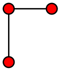

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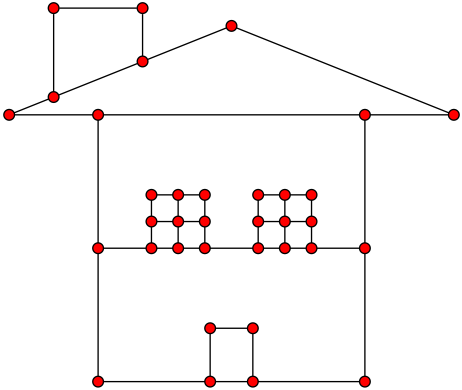

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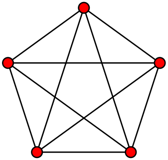
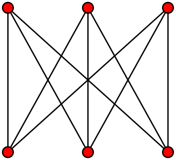
AMAZING FACTS ABOUT PLANAR GRAPHS

Fact

The graphs K_5 and $K_{3,3}$ are not planar.



K_5



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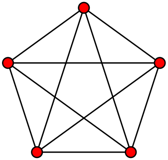

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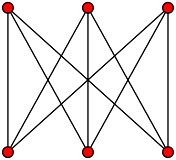

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A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

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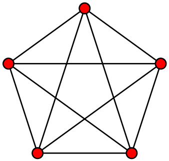

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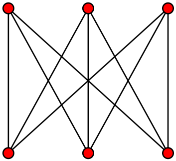

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A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Fáry's theorem

Every planar graph can be drawn in the plane without edges crossing such that every edge is a straight line segment.

EULER'S FORMULA IN ACTION

- *How many faces does a soccer ball have?*
  Suppose that there are P pentagons and H hexagons. Every pentagon is surrounded by five hexagons while every hexagon is surrounded by three pentagons and three hexagons.

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$$\left.\begin{aligned} 3V &= 2E \\ 5P + 6H &= 2E \\ 5P &= 3H \\ V - E + (P + H) &= 2 \end{aligned}\right. \quad \Rightarrow \quad \begin{aligned} P &= 12 \\ H &= 20 \end{aligned}$$

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  Suppose that K_5 is planar and can be drawn in the plane using $V=5$ vertices, $E=10$ edges and F faces.

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- *Why is the graph K_5 not planar?*

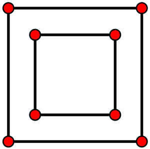
  Suppose that K_5 is planar and can be drawn in the plane using $V = 5$ vertices, $E = 10$ edges and F faces. Then

$$V - E + F = 2 \;\Rightarrow\; F = 7 \qquad \text{and} \qquad 3F \leq 2E \;\Rightarrow\; 21 \leq 20.$$

  This is a contradiction if ever there was one.

TWO PROBLEMS WITH EULER'S FORMULA

- In the example on the right, we find that $V - E + F = 8 - 8 + 3 = 3$.



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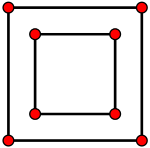
- In the example on the right, we find that $V - E + F = 8 - 8 + 3 = 3$. The reason for the discrepancy is because each face must be a "blob" and have no holes in it.



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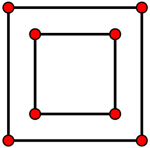

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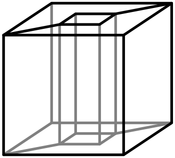

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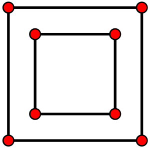

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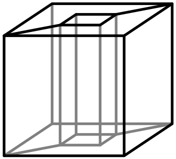

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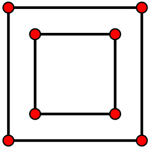

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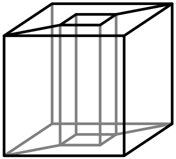

Euler's formula generalised

If a polyhedron with genus g — in other words, one with g holes — has V vertices, E edges and F faces, then

$$V - E + F = 2 - 2g.$$

The Euler characteristic of a surface is the magic number $2 - 2g$.

COUNTING VORTICES

- Is it possible to comb a hairy ball without creating a cowlick?
- Is it possible for the wind on Earth to not have a vortex?
- Is it possible for a smooth vector field on the sphere to not have a zero?

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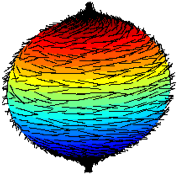
Hairy ball theorem [Brouwer, 1912]

- It is impossible to comb a hairy ball without creating a cowlick.
- It is impossible for the wind on Earth to not have a vortex.
- It is possible for a smooth vector field on the sphere to not have a zero.



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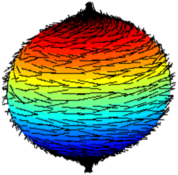

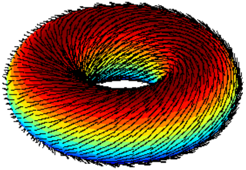

However, it is possible to comb a hairy doughnut without creating a cowlick.

THE POINCARÉ–HOPF THEOREM

Poincaré–Hopf theorem [Poincaré, 1881 and Hopf, 1926]

There is a special way to write a number at every vortex so that the sum of the numbers is equal to the Euler characteristic of the planet.

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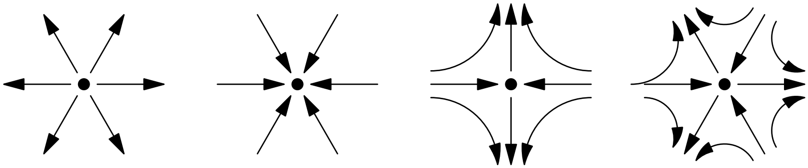
How to count vortices

Walk around the vortex in a small anticlockwise loop, always facing the wind. The index of the vortex is the number of anticlockwise turns that you make.



index $= +1$ index $= +1$ index $= -1$ index $= -2$

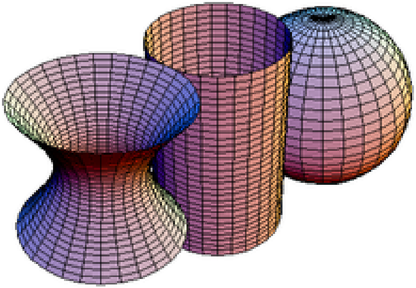
WHY YOU MIGHT BELIEVE POINCARÉ AND HOPF

- The gravy flow on the genus two surface has six vortices.
- The ones at the top and bottom have index $+1$ while the remaining four have index -1.
- These numbers sum up to -2 which is indeed the Euler characteristic of the surface.

CURVATURE FOR DUMMIES

- Naively speaking, curvature measures how "bendy" and "curvy" a surface is at a particular point.
- To find the curvature at a point, you take the smallest circle of best fit and the largest circle of best fit and multiply their radii.
- For example, a sphere of radius R has curvature $\frac{1}{R^2}$.

This picture shows a shape with positive curvature (the sphere), a shape with zero curvature (the cylinder), and a shape with negative curvature (the hyperboloid).



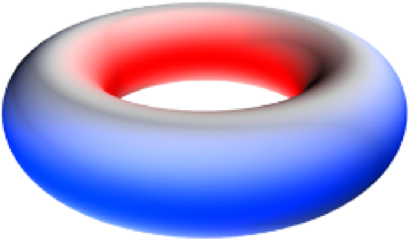
THE GAUSS-BONNET THEOREM

Gauss–Bonnet theorem [Bonnet, 1848]

If you integrate the curvature K over a surface S with respect to the area dA, then you will find that

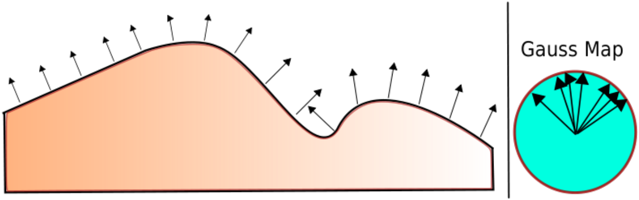
$$\int_S K \; dA = 2\pi\,\chi(S).$$

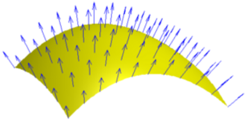
It seems like it could be true — when you deform a surface, you're only spreading out the curvature, never creating or destroying it.



THE GAUSS MAP

- The Gauss map takes a point P on your surface and returns a point $G(P)$ on the unit sphere.
- Consider the normal vector which points directly out of the surface at P.
- Translate this vector to the origin and shrink/expand it until it has length one.
- Then $G(p)$ is the point on the unit sphere which is the endpoint of this vector.





WHY YOU MIGHT BELIEVE GAUSS AND BONNET

Another way to calculate curvature

Draw a tiny, tiny, tiny triangle around P and call it Δ. Then the curvature at P is the ratio $\frac{\text{Area}(G(\Delta))}{\text{Area}(\Delta)}$.

A proof without details

Divide your surface S into many, many, many, tiny, tiny, tiny triangles and call them $\Delta_1, \Delta_2, \Delta_3, \ldots$.

$$\begin{aligned} \int_S K \; dA &= \sum_i K(\Delta) \times \text{Area}(\Delta) = \sum_i \frac{\text{Area}(G(\Delta))}{\text{Area}(\Delta)} \times \text{Area}(\Delta) \\ &= \sum_i \text{Area}(G(\Delta)) = \deg G \times \text{Area}(\text{sphere}) \\ &= 4\pi \; \deg G = 4\pi \, (1 - g) = 2\pi \, \chi(S) \end{aligned}$$

Intuitively, $\deg G$ is the number of times that the surface S is wrapped around the sphere by the map G. You can check that $\deg G = 1 - g$ for your favourite surface S of genus g.

GENERALISING THE EULER CHARACTERISTIC

- *Higher dimensions*

  The higher dimensional analogue of a surface is called a manifold.

  The analogues of vertices, edges and faces are called simplices.

  If a manifold M can be divided into a_k k-dimensional simplices for each k, then

$$\chi(M) = a_0 - a_1 + a_2 - a_3 + \cdots .$$

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- *Bundles*

  A vector field is a choice of vector in every tangent plane of a surface.

  Instead of using the tangent plane, you can use some other collection of planes.

  The resulting object is called a vector bundle.

  The analogue of the Euler characteristic is called the Euler class.

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- *Euler characteristic and cardinality*

  The Euler characteristic is similar to cardinality — it "counts" objects in a set.

  Now we allow a set to have a negative number of objects.

  It turns out that there is also a way to have a fractional number of objects.

  What does it all mean?!

THANKS

If you would like more information, you can

- find the slides at http://www.ms.unimelb.edu.au/~nndo
- email me at normdo@gmail.com
- speak to me at the front of the lecture theatre