VARIATIONS ON VERTICES AND VORTICES

LunchMaths seminar — 18 March 2013

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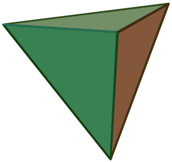
Have you ever tried to count how many panels there are on a soccer ball? Have you ever wondered what the hairy ball theorem is and whether it applies to you? Have you ever thought about how curvaceous a person can be? If you’ve read this much of the abstract, then you should definitely come along to the seminar and learn about the amazing Euler characteristic!

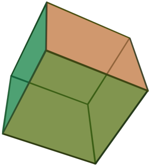
WHAT IS A POLYHEDRON?

Naively speaking, a polyhedron is a shape consisting of vertices, edges, and faces.

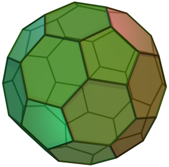
Nice polyhedra



tetrahedron

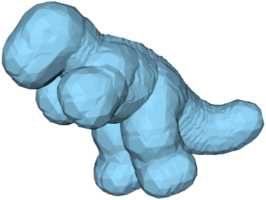


cube



soccer ball

A scary polyhedron



PROPERTIES OF POLYHEDRA

Let's count the number of vertices (V), edges (E), and faces (F).

polyhedron	V	E	F	$V - E + F$
tetrahredron	4	6	4	2
cube	8	12	6	2
soccer ball	60	90	32	2
dinosaur	?	?	?	2

Euler's formula

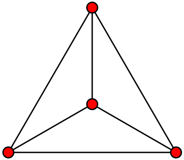
If a polyhedron has V vertices, E edges, and F faces, then

$$V - E + F = 2.$$

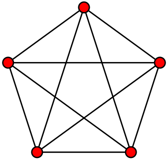
PLANAR GRAPHS

A graph consists of vertices and edges that join vertices in pairs.

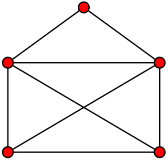
A graph is called planar if you can draw it without edges crossing.



planar



not planar



planar

Euler's formula

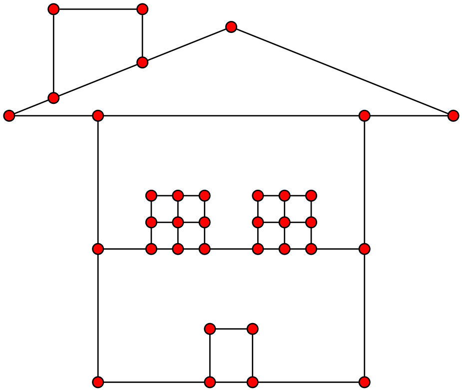
If a planar graph has V vertices, E edges, and divides the plane into F faces including the outside one, then

$$V - E + F = 2.$$

Fact

The vertices and edges of a polyhedron form a planar graph.

WHY YOU MIGHT BELIEVE EULER



$$V - E + F = ?$$

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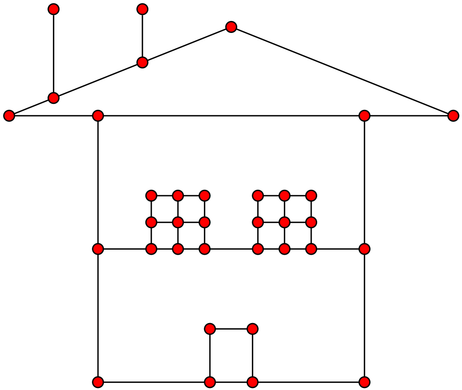

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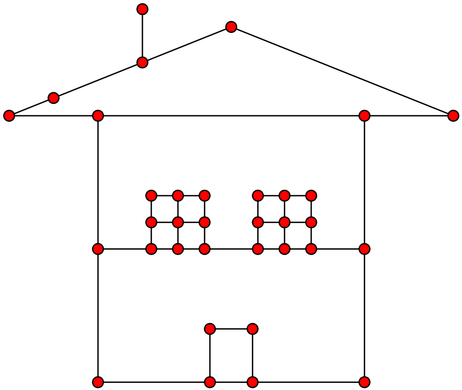

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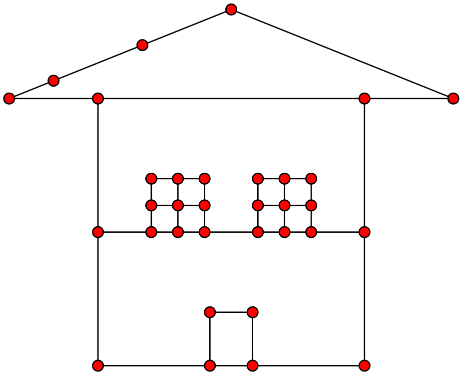

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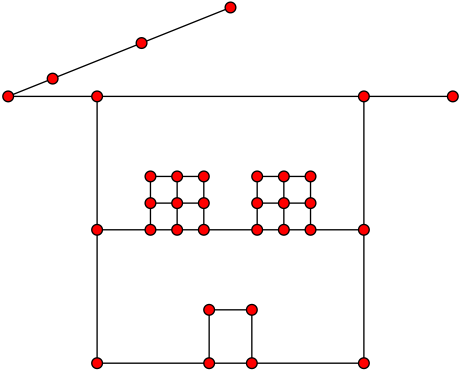

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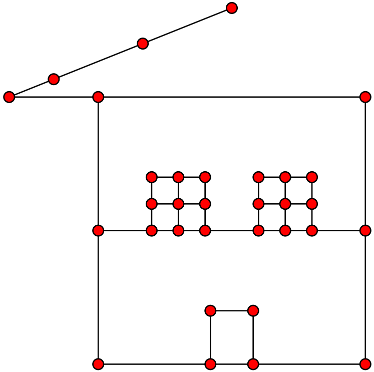

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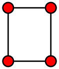

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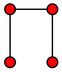

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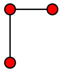

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$$V - E + F = 2$$

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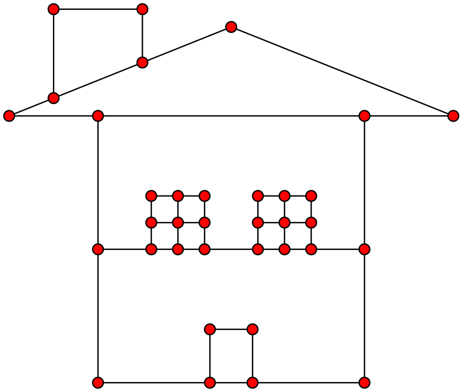
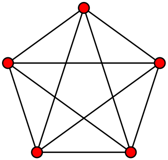


$$V - E + F = 2$$

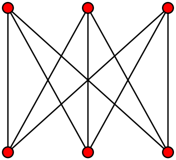
AMAZING FACTS ABOUT PLANAR GRAPHS

Fact

The graphs K_5 and $K_{3,3}$ are not planar.



K_5



$K_{3,3}$

Kuratowski's theorem

A graph is planar if and only if it does not contain "K_5" or "$K_{3,3}$".

Fáry's theorem

Every planar graph can be drawn in the plane without edges crossing such that every edge is a straight line segment.

EULER'S FORMULA IN ACTION

- *How many faces does a soccer ball have?*

  Suppose that there are P pentagons and H hexagons.

  Every pentagon is surrounded by five hexagons while every hexagon is surrounded by three pentagons and three hexagons.

$$\begin{aligned} 3V &= 2E \\ 5P+6H &= 2E \\ 5P &= 3H \\ V-E+(P+H) &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} P &= 12 \\ H &= 20 \end{aligned}$$

- *Why is the graph K_5 not planar?*

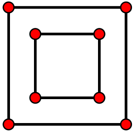
  Suppose that K_5 is planar and can be drawn in the plane using $V=5$ vertices, $E=10$ edges, and F faces. Then

$$V-E+F=2 \Rightarrow F=7 \qquad \text{and} \qquad 3F \leq 2E \Rightarrow 21 \leq 20.$$

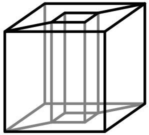
  This is a blatant contradiction!

TWO PROBLEMS WITH EULER'S FORMULA

Here, $V - E + F = 8 - 8 + 3 = 3$.
Each face must be a "2D blob"
without holes.



Here, $V - E + F = 16 - 28 + 12 = 0$.
The polyhedron must enclose a
"3D blob" without holes.



Euler's formula generalised

A polyhedron with genus g — in other words, g holes — satisfies

$$V - E + F = 2 - 2g.$$

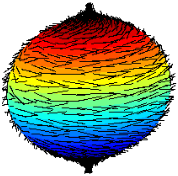
The Euler characteristic of a surface is the magic number $2 - 2g$.

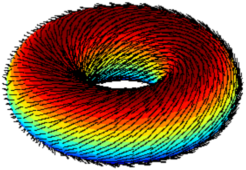
COUNTING VORTICES

- Is it possible to comb a hairy ball without creating a cowlick?
- Is it possible for the wind on Earth to not have a vortex?
- Is it possible for a vector field on the sphere to not have a zero?

Hairy ball theorem [Brouwer, 1912]

It is impossible for a vector field on the sphere to not have a zero.





It is possible to comb a hairy doughnut without creating a cowlick.

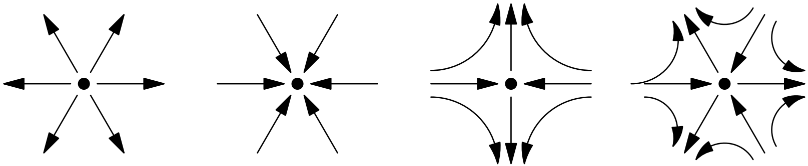
THE POINCARÉ–HOPF THEOREM

Poincaré–Hopf theorem [Poincaré, 1881 and Hopf, 1926]

There is a special way to write a number at every vortex so that the sum of the numbers is equal to the Euler characteristic of the planet.

How to count vortices

Walk around the vortex in a small anticlockwise loop, always facing the wind. The index of the vortex is the number of anticlockwise turns that you make.



index $= +1$ index $= +1$ index $= -1$ index $= -2$

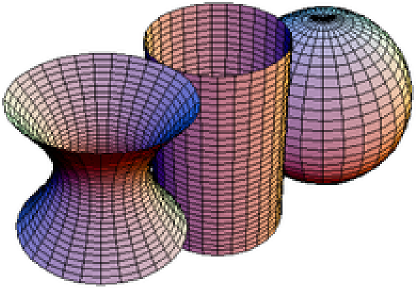
WHY YOU MIGHT BELIEVE POINCARÉ AND HOPF

- The gravy flow on the genus two surface has six vortices.
- The ones at the top and bottom have index $+1$, while the remaining four have index -1.
- These numbers sum up to -2, which is the Euler characteristic.

CURVATURE FOR DUMMIES

- Naively speaking, curvature measures how “bendy” or “curvy” a surface is at a particular point.
- To find the curvature at a point, take the smallest and largest circles of best fit and multiply the inverse of their radii.
- For example, a sphere of radius R has curvature $\frac{1}{R^2}$.

Below are shapes with positive curvature (sphere), zero curvature (cylinder), and negative curvature (hyperboloid).



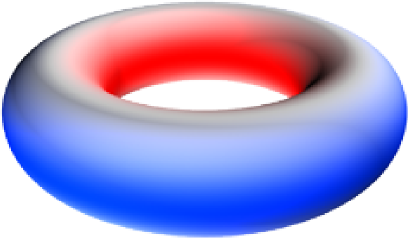
THE GAUSS-BONNET THEOREM

Gauss–Bonnet theorem [Bonnet, 1848]

If you integrate the curvature K over a surface S with respect to the area dA, then you will find that

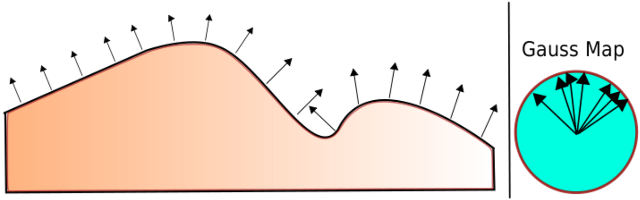
$$\int_S K\, dA = 2\pi\, \chi(S).$$

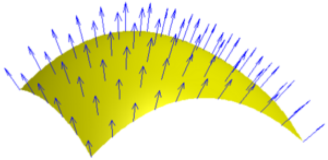
It seems like it could be true — when you deform a surface, you're only spreading out the curvature, never creating or destroying it.



THE GAUSS MAP

- The Gauss map takes a point P on your surface and returns a point $G(P)$ on the unit sphere.
- Consider the vector that points directly out of the surface at P.
- Translate to the origin and shrink/expand until it has length one.
- Then $G(P)$ is the endpoint of this vector.





WHY YOU MIGHT BELIEVE GAUSS AND BONNET

Calculating curvature

Draw a tiny triangle Δ around P. The curvature at P is the ratio

$$\frac{\text{Area } G(\Delta)}{\text{Area } \Delta}.$$

A sketch proof

Divide your surface S into many tiny triangles $\Delta_1, \Delta_2, \Delta_3, \ldots$.

$$\begin{aligned}
\int_S K\, dA &= \sum K(\Delta) \times \text{Area } \Delta = \sum \frac{\text{Area } G(\Delta)}{\text{Area } \Delta} \times \text{Area } \Delta \\
&= \sum \text{Area } G(\Delta) = \deg G \times \text{Area sphere} \\
&= 4\pi \, \deg G = 4\pi\,(1-g) = 2\pi\,\chi(S)
\end{aligned}$$

Intuitively, $\deg G$ is the number of times that the surface S is wrapped around the sphere by the map G.

GENERALISING THE EULER CHARACTERISTIC

- *Higher dimensions*

  The higher dimensional analogue of a surface is a manifold.

  The analogues of vertices, edges, and faces are simplices.

  If a_k is the number of k-dimensional simplices in M, then

$$\chi(M) = a_0 - a_1 + a_2 - a_3 + \cdots .$$

- *Bundles*

  Vector fields choose a vector in each tangent plane of a surface.

  Instead of tangent planes, you can use other collections of planes to create vector bundles.

  The analogue of the Euler characteristic is called the Euler class.

- *Cardinality*

  The Euler characteristic is like cardinality — it “counts” objects.

  Now sets can have a negative number of objects.

  There is also a way to have a fractional number of objects.

  What does it all mean?!

THANKS

If you would like more information, you can

- find the slides at `http://users.monash.edu.au/~normd`
- email me at `norm.do@monash.edu`
- speak to me at `the front of the lecture theatre`