TOPOLOGICAL RECURSION AND THE QUANTUM CURVE FOR MONOTONE HURWITZ NUMBERS

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Take a permutation and count the number of ways to express it as a product of a fixed number of transpositions — you have calculated a Hurwitz number. By adding a mild constraint on such factorisations, one obtains the notion of a monotone Hurwitz number. We have recently shown that the monotone Hurwitz problem fits into the so-called topological recursion/quantum curve paradigm. This talk will attempt to give the flavour of what exactly the previous sentence means.

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Let $H_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ be $\frac{1}{|\boldsymbol{\mu}|!}$ multiplied by the number of tuples $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ of transpositions in $S_{|\boldsymbol{\mu}|}$ such that

- $m = 2g - 2 + n + |\boldsymbol{\mu}|$;
- $\sigma_1 \sigma_2 \cdots \sigma_m$ has labelled cycles of lengths $\mu_1, \mu_2, \ldots, \mu_n$; and
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Secret

Hurwitz numbers count branched covers of $\mathbb{CP}^1$.

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- if $\sigma_i = (a_i \; b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$.

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Secret

Monotone Hurwitz numbers are natural from the viewpoint of

- matrix models (HCIZ integral);
- representation theory (Jucys–Murphy elements); and
- integrability (Toda tau-functions).

Example calculation

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All 24 products produce cycle type $(1, 2)$, so $H_{0,2}(1, 2) = \frac{24}{3!} = 4$.

Only the first 12 products are monotone, so $\vec{H}_{0,2}(1, 2) = \frac{12}{3!} = 2$.

Old results

- **Polynomiality.** There are polynomials $P_{g,n}$ and $\vec{P}_{g,n}$ such that
  - $H_{g,n}(\mu_1, \ldots, \mu_n) = m! \times \prod \frac{\mu_i^{\mu_i}}{\mu_i!} \times P_{g,n}(\mu_1, \ldots, \mu_n)$
  - $\vec{H}_{g,n}(\mu_1, \ldots, \mu_n) = \prod \binom{2\mu_i}{\mu_i} \times \vec{P}_{g,n}(\mu_1, \ldots, \mu_n)$.

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For example, $\vec{P}_{1,2}(\mu_1,\mu_2) = \frac{1}{12}(2\mu_1^2 + 2\mu_2^2 + 2\mu_1\mu_2 - \mu_1 - \mu_2 - 1)$.

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- Cut-and-join recursion. (Monotone) Hurwitz numbers of type (g, n) can be calculated from those of types
  - $(g, n-1)$
  - $(g-1, n+1)$
  - $(g_1, n_1) \times (g_2, n_2)$ for $\begin{array}{l} g_1 + g_2 = g \\ n_1 + n_2 = n + 1. \end{array}$

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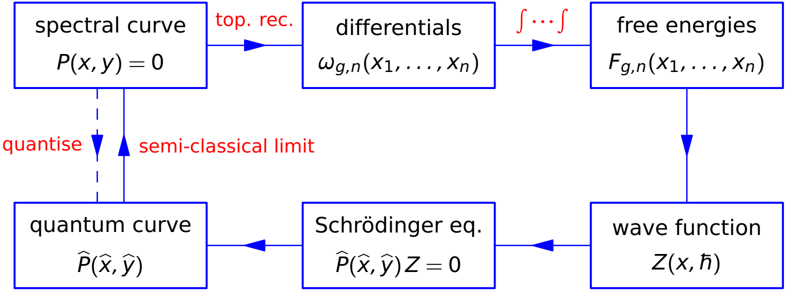
  For example,

$$\mu_1 \vec{H}_{1,2}(\mu_1, \mu_2) = (\mu_1 + \mu_2) \vec{H}_{1,1}(\mu_1 + \mu_2) + \sum_{\alpha+\beta=\mu_1} \alpha\beta \vec{H}_{0,3}(\alpha, \beta, \mu_2)$$
$$+2 \sum_{\alpha+\beta=\mu_1} \alpha\beta \left[\vec{H}_{0,1}(\alpha) \vec{H}_{1,2}(\beta, \mu_2) + \vec{H}_{1,1}(\alpha) \vec{H}_{0,2}(\beta, \mu_2) \right].$$

Topological recursion and quantum curves



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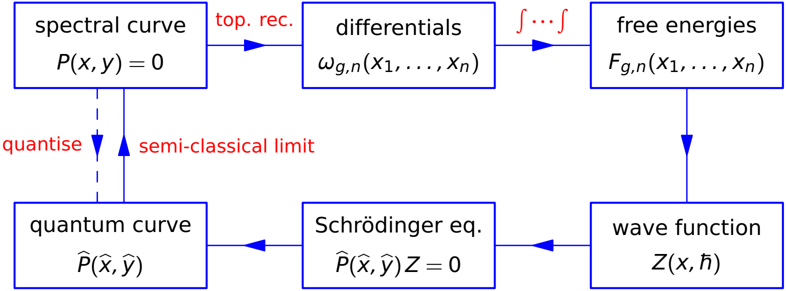

We use the definitions

- $Z(x,\hbar) = \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}F_{g,n}(x,\ldots,x)\right]$
- $\widehat{x}=x$ and $\widehat{y}=-\hbar\frac{\partial}{\partial x}$, which imply $[\widehat{x},\widehat{y}]=\hbar$.

New results

This is joint work with A. Dyer and D. Mathews (`arXiv:1408.3992`).

- The spectral curve $P(x,y) = xy^2 - y + 1 = 0$ yields

$$F_{g,n}(x_1, \ldots, x_n) = \sum_{\boldsymbol{\mu}}^{\infty} \vec{H}_{g,n}(\mu_1, \ldots, \mu_n) \; x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

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- The wave function satisfies

$$Z(x,\hbar) = 1 + \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \begin{matrix} d+m-1 \\ d-1 \end{matrix} \right\} \frac{x^d\, \hbar^{m-d}}{d!}.$$

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- The corresponding quantum curve is $\widehat{P}(\widehat{x}, \widehat{y}) = \widehat{x}\widehat{y}^2 - \widehat{y} + 1$, so

$$x\hbar^2 \frac{\partial^2 Z}{\partial x^2} + \hbar \frac{\partial Z}{\partial x} + Z = 0.$$