SURFACE TILINGS, HURWITZ NUMBERS, AND GROMOV-WITTEN THEORY

Analysis and Geometry Seminar @ Monash University, 14/08/12

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GEOMETRY

Moduli spaces: parametrise geometric objects

different points $\longleftrightarrow$ different objects

nearby points $\longleftrightarrow$ similar objects

Toy example: Moduli space of triangles

$$\mathcal{M}_\Delta = \left\{ (a,b,c) \in \mathbb{R}_+^3 \;\middle|\; \begin{matrix} a+b>c \\ b+c>a \\ c+a>b \end{matrix} \right\} \Big/ S_3$$

Toy question: How many triangles . . .

$$\underbrace{\text{are isosceles,}}_{X_{iso} \subseteq \mathcal{M}_\Delta} \ \underbrace{\text{have a length 5 side,}}_{X_5 \subseteq \mathcal{M}_\Delta} \ \underbrace{\text{and have a length 7 side?}}_{X_7 \subseteq \mathcal{M}_\Delta}$$

Equivalently, what is $|X_{iso} \cap X_5 \cap X_7|$?

$$\| $$

$$\int_{\mathcal{M}_\Delta} X_{iso} \cdot X_5 \cdot X_7$$

(Naive) cohomology:

$$\text{spaces} \longrightarrow H^* \longrightarrow \text{rings}$$

$\{\text{submanifolds}\}/\sim \;\longleftrightarrow$ elements of H^*

union $\longleftrightarrow$ addition

intersection $\longleftrightarrow$ multiplication

MODULI SPACES OF CURVES

Surface topology:  . . .

$g=0$ $\quad$ $g=1$ $\quad$ $g=2$

Surface geometry:

SURFACES

$$\begin{Bmatrix}\text{algebraic}\\ \text{curves}\end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix}\text{Riemann}\\ \text{surfaces}\end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix}\text{hyperbolic}\\ \text{surfaces}\end{Bmatrix}$$

$\mathcal{M}_{g,n} = \begin{Bmatrix}\text{genus } g \text{ smooth algebraic}\\ \text{curves with } n \text{ labelled points}\end{Bmatrix}$ e.g.  $\in \mathcal{M}_{2,3}$

$\overline{\mathcal{M}}_{g,n} = \begin{Bmatrix}\text{genus } g \text{ stable algebraic}\\ \text{curves with } n \text{ labelled points}\end{Bmatrix}$ e.g.  $\in \overline{\mathcal{M}}_{2,3}$

stable = allow nodes + finitely many automorphisms

Facts:

- $\dim \overline{\mathcal{M}}_{g,n} = 2(3g-3+n)$
- $\overline{\mathcal{M}}_{g,n}$ is an orbifold, so intersection numbers are rational
- $\overline{\mathcal{M}}_{g,n}$ is VERY complicated
- natural cohomology classes: $\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n})$

  $\lambda_i \in H^{2i}(\overline{\mathcal{M}}_{g,n})$ for $i = 0, 1, 2, \ldots$

Gromov-Witten theory: finding intersection numbers for $\overline{\mathcal{M}}_{g,n}$ and friends

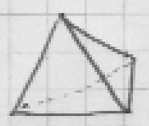
AIM: COUNT SURFACES

SURFACE TILINGS

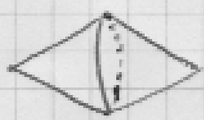
Question: Given labelled polygons with $b_1, b_2, \ldots, b_n$ sides, how many ways are there to make a genus g surface?

Answer: $N_{g,n}(b_1, b_2, \ldots, b_n)$

Example: $N_{0,4}(3,3,3,3) = 8$

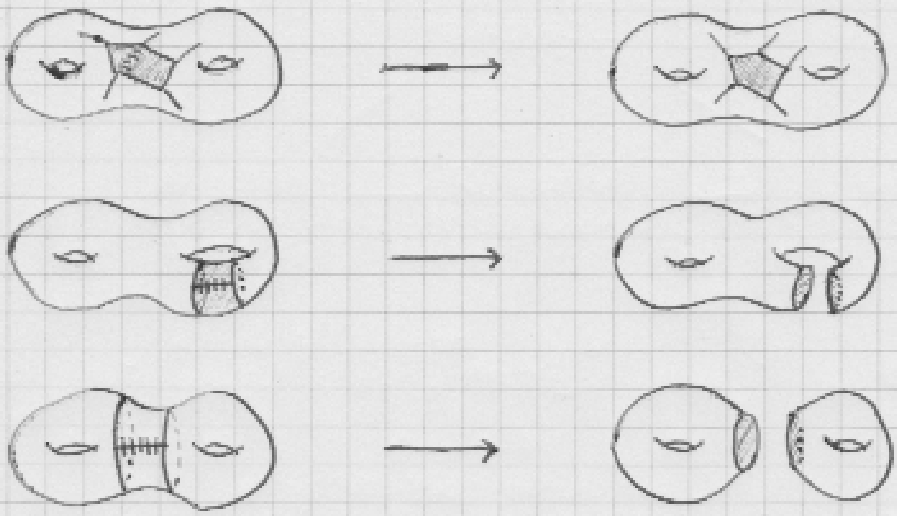


2 labellings



6 labellings

Recursion: Remove an edge



So $N_{g,n}$ depends on $N_{g,n-1}$

$N_{g-1,n+1}$

$N_{g_1,n_1} \times N_{g_2,n_2}$ for $\begin{matrix} g_1+g_2=g \\ n_1+n_2=n+1 \end{matrix}$ } topological recursion

Corollary: $N_{g,n}$ is a quasi-polynomial in $b_1^2, b_2^2, \ldots, b_n^2$ of degree $3g-3+n$

Theorem:

- For $a_1+a_2+\cdots+a_n = 3g-3+n$,

$$\left[b_1^{2a_1} b_2^{2a_2} \cdots b_n^{2a_n} \right] N_{g,n} = \frac{1}{2^{5g-6+2n} a_1! a_2! \cdots a_n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}$$

- $N_{g,n}(0,0,\ldots,0) = \chi(\mathcal{M}_{g,n})$

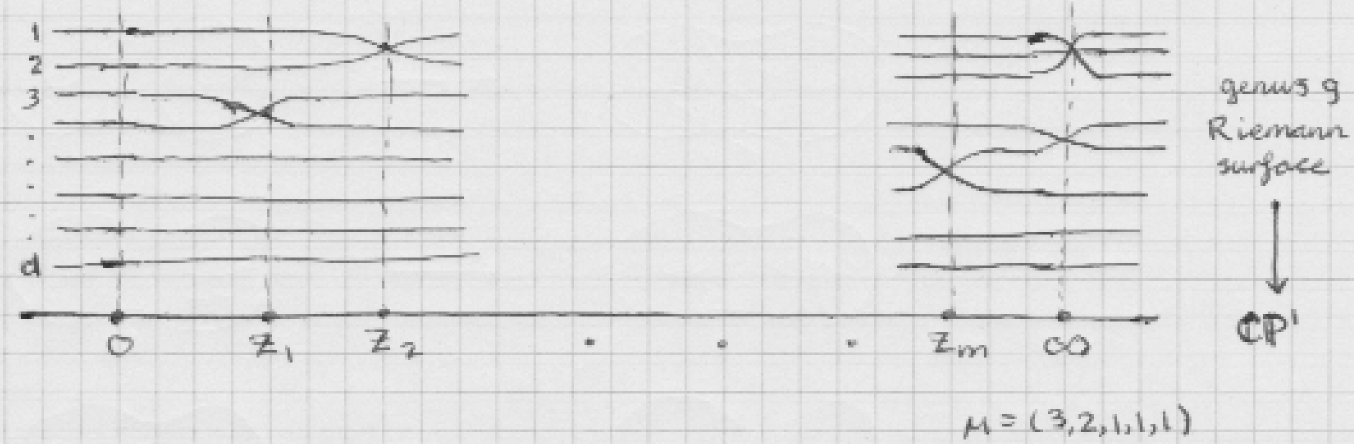
HURWITZ NUMBERS

Question: How many maps are there from a genus g Riemann surface to $\mathbb{CP}^1$?

Hurwitz numbers: Let $g \geq 0$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be a partition of d.

$H_{g;\mu}$ = # genus g branched coverings of $\mathbb{CP}^1$ with branching profile μ over ∞ and simple branching at m fixed points

Riemann-Hurwitz formula $\Rightarrow$ $m = 2g - 2 + \sum \mu_i + n$



By considering the permutations of the d sheets of the branched cover formed by walking from 0, around z_i, and then back to 0, we get the following.

$$H_{g;\mu} = \frac{1}{d!} \# \text{ tuples } (\sigma_1, \sigma_2, \ldots, \sigma_m) \text{ of transpositions in } S_d$$

where $\sigma_1 \sigma_2 \cdots \sigma_m$ has cycle type μ and

$\sigma_1, \sigma_2, \ldots, \sigma_m$ generate S_d

Recursion: Cut and join

Hurwitz numbers of type (g, n) depend on

- type $(g, n-1)$
- type $(g-1, n+1)$
- type $(g_1, n_1) \times$ type (g_2, n_2) where $g_1 + g_2 = g$, $n_1 + n_2 = n+1$

} topological recursion

ELSV formula:

$$H_{g;\mu} = \frac{m!}{\#\mathrm{Aut}\,\mu} \prod_{k=1}^{n} \frac{\mu_k^{\mu_k}}{\mu_k!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{(1 - \mu_1 \psi_1) \cdots (1 - \mu_n \psi_n)}$$