Norman Do

Tiles have been used in art and architecture since the dawn of civilisation. Toddlers grapple with tiling problems when they pack away their wooden blocks and home renovators encounter similar conundrums in the bathroom. However, rather than being a frivolous pastime, mathematicians have found the art of tiling to be brimming with amazing results. In this seminar, we will discover the colourful world of tiles, learn about faulty tilings, unlock the secrets of the Aztec diamond, and discuss a sequence of numbers which (I bet) grows faster than any you have ever imagined!

Dominoes on a checkerboard

Questions

- Can you tile an 8×8 checkerboard with dominoes?

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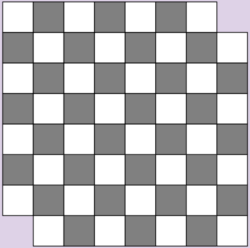

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- Can you tile an 8×8 checkerboard with dominoes?
- Can you tile the checkerboard if one corner is removed?
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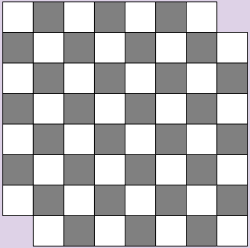
- Can you always tile the checkerboard if one white square and one black square are removed?



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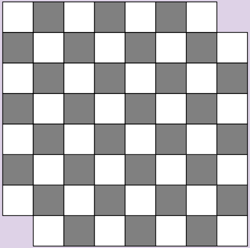

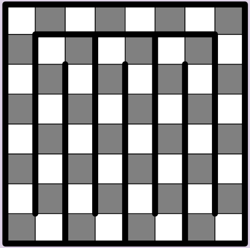

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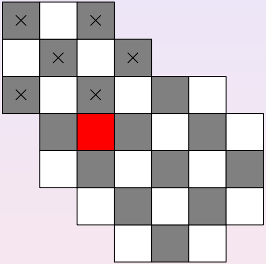
When can you tile a mutilated checkerboard with dominoes?



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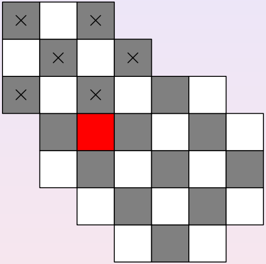

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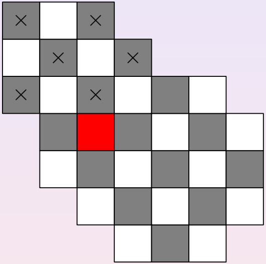

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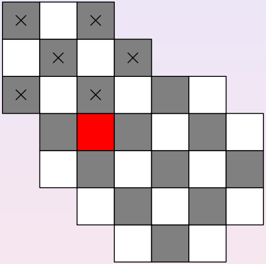

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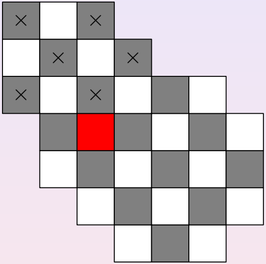

- Black squares are men and white squares are women.
- We want to marry everyone off to their neighbour.
- We need to have gender balance. . . but we also need every group of men to have enough women to marry.

Answer

If every group of men has enough women to marry and vice versa, then you can tile the mutilated checkerboard with dominoes.

Dominoes on a checkerboard

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How many ways can you tile an 8×8 checkerboard with dominoes?

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Theorem (Fisher–Temperley and Kasteleyn, 1961)

The number of tilings of a $2m \times 2n$ checkerboard with dominoes is

$$\prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos^2 \frac{j\pi}{2m+1} + 4\cos^2 \frac{k\pi}{2n+1} \right).$$

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$$\begin{array}{cccc}
\left(4\cos^2 \frac{\pi}{9} + 4\cos^2 \frac{\pi}{9}\right) & \left(4\cos^2 \frac{\pi}{9} + 4\cos^2 \frac{2\pi}{9}\right) & \left(4\cos^2 \frac{\pi}{9} + 4\cos^2 \frac{3\pi}{9}\right) & \left(4\cos^2 \frac{\pi}{9} + 4\cos^2 \frac{4\pi}{9}\right) \\
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\end{array}$$

$$= 7.06417777 \times 5.87938524 \times 4.53208889 \times \ldots = 12988816 = 3604^2$$

Rectangles on a checkerboard

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When can you tile an $m \times n$ checkerboard with $a \times b$ rectangles?

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- Can you tile a 12×15 checkerboard with 4×7 rectangles?
  NO. . . the area of one tile does not divide the area of the board.

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- Can you tile a 17×28 checkerboard with 4×7 rectangles?
  NO. . . you can't even tile the first column.

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- Can you tile a 17×28 checkerboard with 4×7 rectangles?
  NO. . . you can't even tile the first column.
- Can you tile an 14×18 checkerboard with 4×7 rectangles?
  NO. . . you can't even tile it with 4×1 rectangles.

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- Can you tile an 14×18 checkerboard with 4×7 rectangles?
  NO. . . you can't even tile it with 4×1 rectangles.

We will prove this by using COLOURS!

Rectangles on a checkerboard

Every 4×1 rectangle covers one square of each colour. On a 14×18 checkerboard, there are fewer squares of colour 1 than of colour 2.

1	2	3	4	1	2	3	4	1	2	3	4	⋯
2	3	4	1	2	3	4	1	2	3	4	1	⋯
3	4	1	2	3	4	1	2	3	4	1	2	⋯
4	1	2	3	4	1	2	3	4	1	2	3	⋯
1	2	3	4	1	2	3	4	1	2	3	4	⋯
2	3	4	1	2	3	4	1	2	3	4	1	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

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4	1	2	3	4	1	2	3	4	1	2	3	⋯
1	2	3	4	1	2	3	4	1	2	3	4	⋯
2	3	4	1	2	3	4	1	2	3	4	1	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Answer

Let a and b be relatively prime positive integers. A tiling of an $m \times n$ rectangles with $a \times b$ rectangles exists if and only if

- both m and n can be written as $xa + yb$, where x and y are non-negative integers; and
- either m or n is divisible by a, and either m or n is divisible by b.

Faulty tilings

All Soviet Union Mathematical Olympiad 1963

A 6×6 checkerboard is tiled with dominoes. Prove that you can cut the board with a line which does not pass through any domino.

Faulty tilings

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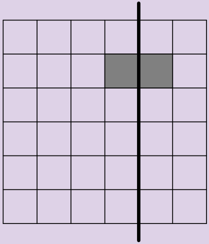
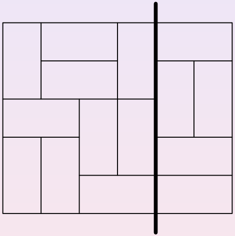
Proof.

- To obtain a contradiction, we suppose otherwise.
- There are 10 lines — 5 horizontal and 5 vertical — each of which must be crossed by at least 1 domino.
- But each of these lines must actually be crossed by at least 2 dominoes. . . see the diagram on the right.
- So there must be at least 20 dominoes, which gives the desired contradiction! □

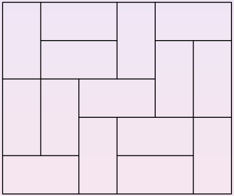


Faulty tilings

- Consider a tiling of a checkerboard by rectangles. If you can cut the board with a line which does not pass through any tile, then the tiling is called faulty. Otherwise, the tiling is called faultless.



faulty



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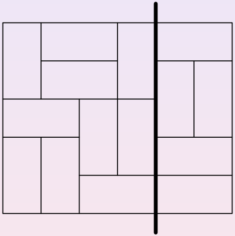

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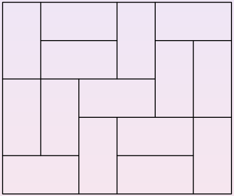


faultless

- We have already shown that every domino tiling of a 6×6 checkerboard is faulty.

Faulty tilings

Question

When can you find a faultless tiling of an $m \times n$ checkerboard with $a \times b$ rectangles?

Faulty tilings

Question

When can you find a faultless tiling of an $m \times n$ checkerboard with $a \times b$ rectangles?

Answer

Assume that a and b are relatively prime. You can find a faultless tiling of an $m \times n$ checkerboard with $a \times b$ rectangles if and only if

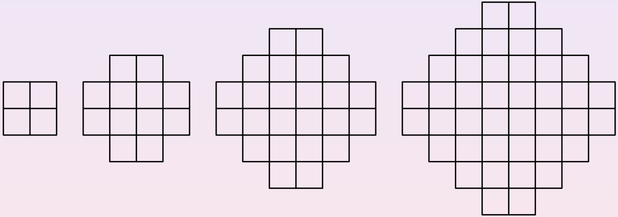
- either m or n is divisible by a, and either m or n is divisible by b;
- both m and n can be expressed as $xa + yb$ in at least two ways, where x and y are positive integers; and
- if the tiles are dominoes, the checkerboard is not 6×6.

Aztec diamonds

Question

How many ways can you tile an Aztec diamond with dominoes?

Here are the Aztec diamonds $AZ(1), AZ(2), AZ(3)$ and $AZ(4)$.



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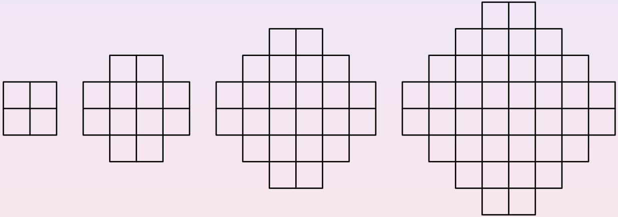
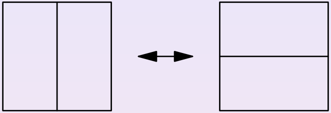


Answer

The number of ways to tile the Aztec diamond $AZ(n)$ is $2^{n(n+1)/2}$.

Aztec diamonds

A domino flip is a move which takes two adjacent dominoes in a tiling and rotates them by 90°.



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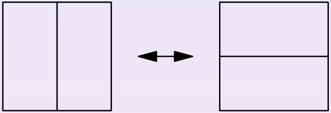
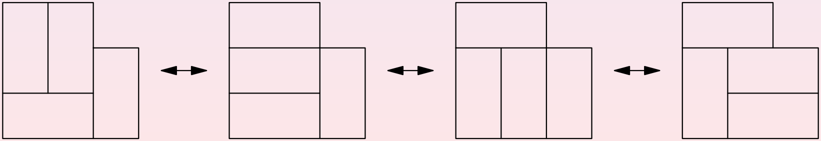


Domino flipping theorem

Any two domino tilings of a shape without holes are related by domino flipping.



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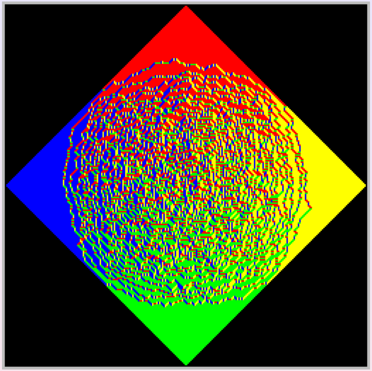

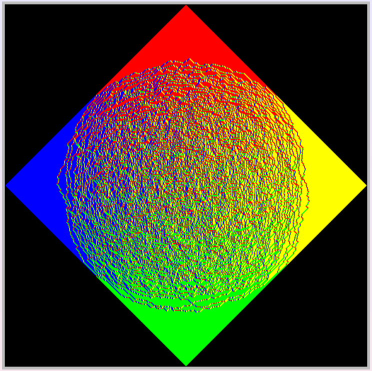

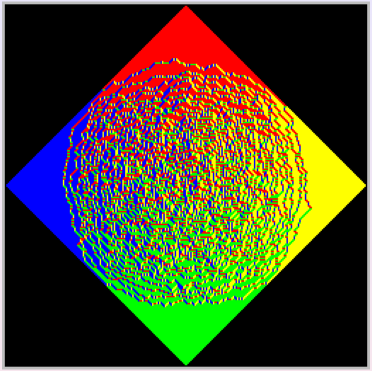

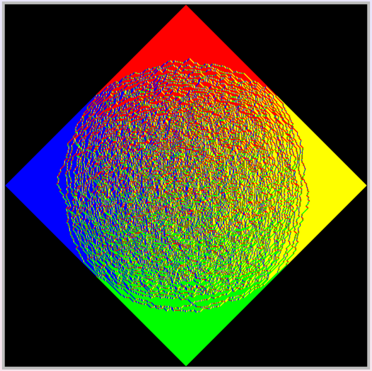

Arctic circle theorem

As n approaches infinity, the disorded region of a random domino tiling of the Aztec diamond $AZ(n)$ will approach a circle.

Tiling a square with similar rectangles

Question

For what values of r can you tile a square with rectangles which are similar to a $1 \times r$ rectangle?

Tiling a square with similar rectangles

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- Call the number r happy if you can tile a square with rectangles which are similar to a $1 \times r$ rectangle.

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- Call the number r happy if you can tile a square with rectangles which are similar to a $1 \times r$ rectangle.
- If $r = \frac{a}{b}$ is a positive rational number, then you simply need to tile a square with $a \times b$ rectangles. Since this is possible, all rational numbers are happy...but are all happy numbers rational?

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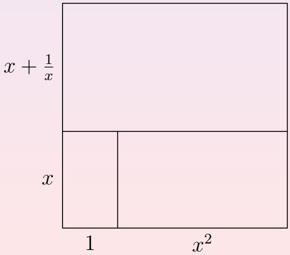
- If $r = \frac{a}{b}$ is a positive rational number, then you simply need to tile a square with $a \times b$ rectangles. Since this is possible, all rational numbers are happy. . . but are all happy numbers rational?
- No, it is possible for irrational numbers to be happy too!

$$x^2 + 1 = 2x + \frac{1}{x}$$

$$x^3 - 2x^2 + x - 1 = 0$$

$x = 1.75488$ or $0.122561 \pm 0.744862i$



Tiling a square with similar rectangles

Polynomials and algebraic numbers

- A number is called algebraic if it is the root of a non-zero polynomial with integer coefficients.

  For example, $\sqrt{2}$ and $1 + i$ are algebraic but π and e are not.

- Every algebraic number is the root of infinitely many non-zero polynomials with integer coefficients.

  For example, $\sqrt{2}$ is the root of the polynomials $x^2 - 2$, $-7x^2 + 14$ and $(x^2 - 2)(x^3 + 3x + 1)$.

- The most efficient one is called the minimal polynomial.

  For example, the minimal polynomial of $\sqrt{2}$ is $x^2 - 2$.

- If two algebraic numbers are roots of the same minimal polynomial, then we say that they are friends. (The technical term is "Galois conjugates".)

  For example $\sqrt{2}$ and $-\sqrt{2}$ are friends.

Tiling a square with similar rectangles

Answer

The number r is happy if and only if it is a positive real algebraic number whose friends all have positive real part.

Tiling a square with similar rectangles

Answer

The number r is happy if and only if it is a positive real algebraic number whose friends all have positive real part.

- The number $\sqrt{2}$ is not happy, because the polynomial $x^2 - 2$ also has the root $-\sqrt{2}$.
- In fact, $\frac{a}{b} + \sqrt{2}$ is happy if and only if $\frac{a}{b} > \sqrt{2}$. This is because the minimal polynomial of $\frac{a}{b} + \sqrt{2}$ is

$$b^2x^2 - 2abx + a^2 - 2b^2,$$

which also has the root $\frac{a}{b} - \sqrt{2}$.

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  which also has the root $\frac{a}{b} - \sqrt{2}$.

Challenge

Try to tile a square with rectangles which are similar to a $(\frac{3}{2} + \sqrt{2}) \times 1$ rectangle.

Tiling a square with infinitely many rectangles

Consider the equation

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots=1.$$

This suggests the following…

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This suggests the following. . .

Question

Can you tile a unit square with one tile of size $1\times\frac{1}{2}$, one tile of size $\frac{1}{2}\times\frac{1}{3}$, one tile of size $\frac{1}{3}\times\frac{1}{4}$, and so on?

Tiling a square with infinitely many rectangles

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- Amazingly, no one knows the answer to this question!
- However, someone has managed to squeeze these rectangles into a square of side length 1.000000001!

Polyominoes in the plane

A polyomino is a shape obtained by gluing unit squares together along their edges.

The who-can-tile-lots-but-not-all game

- I give you n unit squares and you construct a set of polyominoes.
- If it is possible to tile the whole plane with tiles of these shapes, then you lose.
- If it is not possible to tile the whole plane with tiles of these shapes, then you win L dollars where L is the side length of the largest square you can cover with tiles of these shapes.

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Define $L(n)$ to be the largest number of dollars you can win.

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- I give you n unit squares and you construct a set of polyominoes.
- If it is possible to tile the whole plane with tiles of these shapes, then you lose.
- If it is not possible to tile the whole plane with tiles of these shapes, then you win L dollars where L is the side length of the largest square you can cover with tiles of these shapes.

Define $L(n)$ to be the largest number of dollars you can win.

Theorem

The sequence $L(1), L(2), L(3), \ldots$ grows quicker than any sequence which can be output by a computer program.

Consequences

- A computer program can output the sequence

$$n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}, \text{ where the tower is } n \text{ high.}$$

The first three terms are 1, 4, 7625597484987, and the fourth has over 8×10^{153} digits. This sequence pales in comparison to many other sequences which a computer program can output, all of which pale in comparison to the sequence $L(1), L(2), L(3), \ldots$.

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- It is impossible to write a computer program to calculate $L(n)$.

Polyominoes in the plane

Consequences

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$$n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}, \text{ where the tower is } n \text{ high.}$$

  The first three terms are 1, 4, 7625597484987, and the fourth has over 8×10^{153} digits. This sequence pales in comparison to many other sequences which a computer program can output, all of which pale in comparison to the sequence $L(1), L(2), L(3), \ldots$.
- It is impossible to write a computer program to calculate $L(n)$.
- There probably exists a set of polyominoes constructed from no more than 100 squares of side length one centimetre which satisfies the following conditions.

  It is impossible to tile the whole plane with them, but it is possible to tile a region which covers Australia.

For more information

- Read my article at

  `http://www.austms.org.au/Publ/Gazette/`

- Email me at

  `norm@ms.unimelb.edu.au`

- Speak to me at

  `the front of the lecture theatre`