ORBIFOLD HURWITZ NUMBERS & TOPOLOGICAL RECURSION

Sydney GTA Seminar – 12/02/13

Norm Do

COUNTING SURFACES

- surface tilings
- volumes of moduli spaces $\overline{\mathcal{M}}_{g,n}$
- $\boxed{\text{branched covers}}$
- Gromov – Witten theory
- theoretical physics

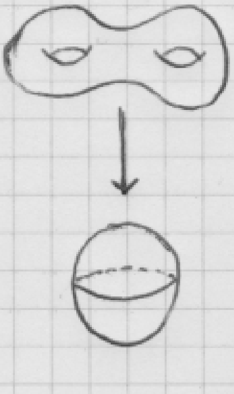
SIMPLE HURWITZ NUMBERS

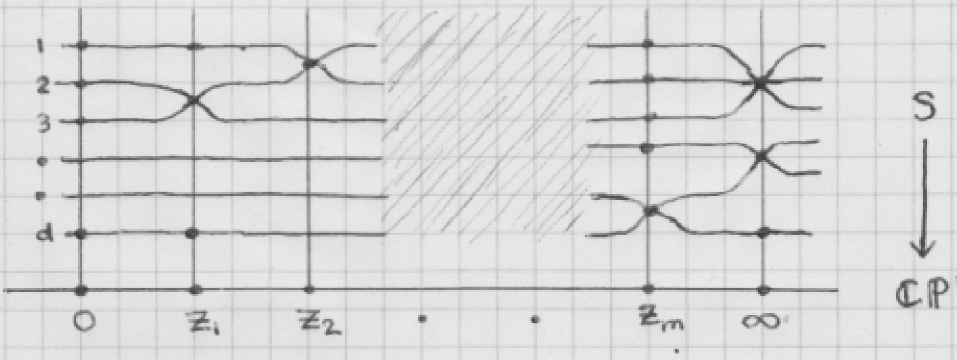
$H_{g;\mu}$ = # genus g branched covers of $\mathbb{CP}^1$ with

integer partition

- ramification profile μ over ∞
- simple ramification at m fixed points
- no other branching

Riemann-Hurwitz $\Rightarrow$ $m = 2g - 2 + n + \Sigma \mu_i$





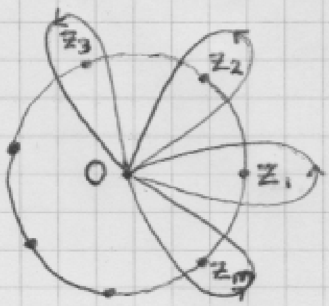
FOUR WAYS TO COMPUTE

① Monodromy

possibly disconnected

$H^\bullet_{g;\mu} = \frac{1}{d!}$ # $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ transpositions in S_d with

$\sigma_1 \sigma_2 \cdots \sigma_m$ of cycle type μ



② Burnside formula

$$H^{\bullet}_{g;\mu} = \frac{1}{d!^2} \sum_{\chi} \frac{[\chi(C_{\text{transposition}})]^m \, \chi(C_\mu)}{[\chi(C_{\text{identity}})]^{m-1}}$$

conjugacy class in S_d

irreducible character of S_d

③ ELSV formula [2001]

$$H_{g;\mu} = \underbrace{\frac{m!}{\#\mathrm{Aut}\,\mu} \times \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!}}_{\text{combinatorial factor}} \times \underbrace{\sum_{\substack{a_1+\cdots+a_n+k \\ = 3g-3+n}} (-1)^k \left[\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_k\right] \mu_1^{a_1} \cdots \mu_n^{a_n}}_{\text{polynomial in } \mu_1, \ldots, \mu_n}$$

④ Cut-and-join

Let $H_g(\mu) = H_{g;\mu} \times \frac{\#\mathrm{Aut}\,\mu}{m!}$ and $S = \{1, 2, \ldots, n\}$.

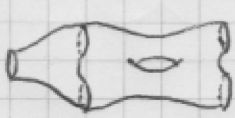
$$m\, H_g(\mu) = \sum_{i<j} H_g(\mu_{S\setminus\{i,j\}}, \mu_i + \mu_j)$$

$$+ \sum_i \sum_{\alpha+\beta=\mu_i} \frac{\alpha\beta}{2} \Big[H_{g-1}(\mu_{S\setminus\{i\}}, \alpha, \beta) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}} H_{g_1}(\mu_I, \alpha)\, H_{g_2}(\mu_J, \beta) \Big]$$

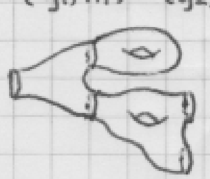
Moral: (g,n) depends on

$(g, n-1)$

$(g-1, n+1)$



$(g_1, n_1) \times (g_2, n_2)$ for $g_1 + g_2 = g$, $n_1 + n_2 = n+1$



TOPOLOGICAL RECURSION [Eynard - Orantin, 2007]

$$\begin{matrix} x(z) \\ y(z) \end{matrix} \longrightarrow \boxed{\text{EO}} \longrightarrow \begin{matrix} \omega_{g,n}(z_1, z_2, \ldots, z_n) \\ \text{for } g \geq 0 \text{ and } n \geq 1 \end{matrix}$$

$$\omega_{g,n}(z_S) = \sum_{dx(a)=0} \operatorname*{Res}_{z=a} K(z_1, z) \Big[\omega_{g-1,n+1}(z, s_a(z), z_{S\setminus\{i\}})$$

$$+ \hat{\sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}}} \omega_{g_1,|I|+1}(z, z_I)\, \omega_{g_2,|J|+1}(s_a(z), z_J) \Big]$$

Same moral!

BOUCHARD–MARIÑO CONJECTURE [2008]

$$\begin{matrix} x(z) = z\exp(-z) \\ y(z) = z \end{matrix} \longrightarrow \boxed{\mathrm{EO}} \longrightarrow \frac{\omega_{g,n}(x_1,\ldots,x_n)}{dx_1 \cdots dx_n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} H_g(\mu) \prod_{i=1}^{n} \mu_i x_i^{\mu_i - 1}$$

- Inspired by topological string theory
- Proven by (Borot)–Eynard–Mulase–Safnuk in 2009

ORBIFOLD HURWITZ NUMBERS

$H^{[a]}_{g;\mu}$ = # genus g branched covers of $\mathbb{CP}^1$ with

- ramification profile μ over ∞
- simple branching at m fixed points
- ramification profile $(a, a, \ldots, a)$ over 0
- no other branching

Riemann–Hurwitz $\Rightarrow$ $m = 2g - 2 + n + \frac{\sum \mu_i}{a}$

We still have

① monodromy description

② Burnside formula

③ moduli space formula [Johnson–Pandharipande–Tseng, 2011]

$$H^{[a]}_{g;\mu} = \frac{m!}{\#\mathrm{Aut}\,\mu}\, a^{1-g+\sum\{\mu_i/a\}} \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \mu_i/a \rfloor}}{\lfloor \mu_i/a \rfloor !} \sum_{\substack{a_1+\cdots+a_n+k \\ = 3g-3+n}} (-a)^k \left[\int_{\overline{\mathcal{M}}_{g,[-\mu]}(B\mathbb{Z}_a)} \overline{\psi}_1^{a_1} \cdots \overline{\psi}_n^{a_n} \overline{\lambda}_k \right] \times \mu_1^{a_1} \cdots \mu_n^{a_n}$$

④ cut-and-join

These numbers feature in the Gromov–Witten theory of orbifolds.

Theorem: [Do–Leigh–Norbury, 2012]

$$\begin{matrix} x(z) = z\exp(-z^a) \\ y(z) = z^a \end{matrix} \longrightarrow \boxed{\mathrm{EO}} \longrightarrow \frac{\omega_{g,n}(x_1,\ldots,x_n)}{dx_1 \cdots dx_n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} H^{[a]}_{g;\mu} \prod_{i=1}^{n} \mu_i x_i^{\mu_i - 1}$$

STRUCTURE OF PROOF

cut-and-join

$\big\downarrow$ multiply by $x_1^{\mu_1} \cdots x_n^{\mu_n}$, sum over $\mu_1, \ldots, \mu_n$

PDE for generating functions

$\big\downarrow$ express as sum over principal parts w.r.t z_1

almost there

$\big\downarrow$ take symmetric part w.r.t. local involutions

topological recursion

$$PP[f(z)]_a = \operatorname*{Res}_{w=a} \frac{f(w)\,dw}{z-w}$$

$$\omega = \sum_a PP[\omega]_a$$

APPLICATIONS OF TOPOLOGICAL RECURSION

- matrix models (e.g. $\int_{\mathcal{H}_N} f(M) \exp[-\frac{1}{2} \operatorname{tr} M^2]\, dM$
- surface tilings
- intersection theory on

  $\overline{\mathcal{M}}_{g,n}$ = moduli space of genus g Riemann surfaces with n labelled points
- Gromov-Witten theory (enumerative geometry of maps from Riemann surfaces to varieties)
- knot theory

  A-polynomial $\rightarrow \boxed{\text{EO}} \rightarrow$ volume, Jones polynomials, Chern-Simons theory, ...