LATTICE POINTS IN MODULI SPACES OF CURVES

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MODULI SPACES

surfaces
(g = genus, n = #boundaries)

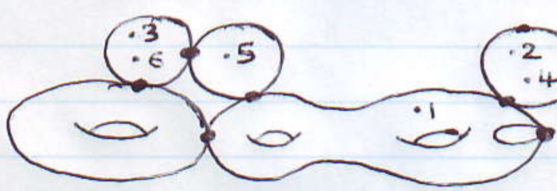
algebraic curves | Riemann surfaces | hyperbolic surfaces

$$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ stable curves with } n \\ \text{smooth points labelled } 1, 2, \ldots, n \end{array} \right\}$$

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$$

STABLE CURVES

* allow nodes
* no components with $2 - 2g - n < 0$
  $(g, n) \neq (0,0), (0,1), (0,2), (1,0)$

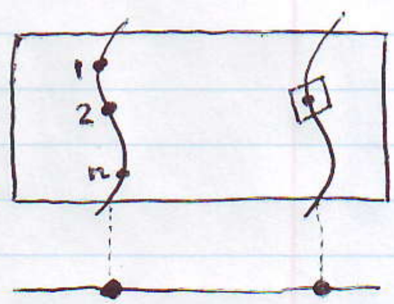


$\in \overline{\mathcal{M}}_{5,6}$

INTERSECTION THEORY

$\mathcal{L}$ = cotangents to fibres

$$\begin{array}{c} \overline{\mathcal{M}}_{g,n+1} \\ \sigma_1, \sigma_2, \ldots, \sigma_n \uparrow \;\; \downarrow \pi \text{ forgets } n+1 \\ \overline{\mathcal{M}}_{g,n} \end{array}$$



For $k = 1, 2, \ldots, n$, $\psi_k = c_1(\sigma_k^* \mathcal{L}) \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$.

Goal: Compute $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n} \in \mathbb{Q}$ where $|\alpha| = 3g-3+n$.

WITTEN'S CONJECTURE (1991)

a natural generating function for these numbers = a KdV tau function

NOT Algebraic Geometry (volumes)	Algebraic Geometry (Hurwitz numbers)
Kontsevich (1992)	Okounkov / Pandharipande (2001)
Mirzakhani (2003)	Kazarian / Lando (2005)
	Kim / Liu (2006)

Hurwitz numbers count

* covers of $\mathbb{P}^1$
* factorisations in S_n
* maps on surfaces

RIBBON GRAPHS

Strebel's theorem: Using quadratic differentials...

* Riemann surface C		* cell decomposition of C
* points labelled $1, 2, \ldots, n$	$\xleftrightarrow{\text{one-to-one}}$	* one face for each point
* positive $r_1, r_2, \ldots, r_n$		* positive edge lengths where boundary of face k has length r_k
		metric ribbon graph

Corollary: For all $r_1, r_2, \ldots, r_n \in \mathbb{R}_+$

$$\mathcal{M}_{g,n} \cong \left\{ \begin{array}{l} \text{metric ribbon graphs with genus } g,\ n \text{ faces} \\ \text{and boundary of face } k \text{ has length } r_k \end{array} \right\}$$

MIRZAKHANI'S PROOF

$$\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n) = \left\{ \begin{array}{l} \text{genus } g \text{ hyperbolic surfaces with} \\ \text{boundaries of lengths } L_1, L_2, \ldots, L_n \end{array} \right\}$$

* FORMULA

$$V_{g,n}(L_1, L_2, \ldots, L_n) = \sum_{|\alpha|+m=3g-3+n} \frac{(2\pi)^{2m} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n} \kappa_1^m}{2^{3g-3+n}\, \alpha_1!\, \alpha_2! \cdots \alpha_n!\, m!} L_1^{2\alpha_1} L_2^{2\alpha_2} \cdots L_n^{2\alpha_n}$$

* RECURSION

$V_{g,n}$ depends on $V_{g,n-1}$

$V_{g-1,n+1}$

$V_{g_1,n_1} \times V_{g_2,n_2}$ for $\begin{array}{l} g_1 + g_2 = g \\ n_1 + n_2 = n+1 \end{array}$

* FORMULA + RECURSION = WITTEN'S CONJECTURE

NORBURY'S IDEA

VOLUME ≈ # LATTICE POINTS

Goal: count metric ribbon graphs with integer edge lengths

$N_{g,n}(b_1, b_2, \ldots, b_n)$ = # metric ribbon graphs with genus g, n faces, integer edge lengths and boundary of face k has length r_k

Note:

* If $N_{g,n} \neq 0$, then $b_1 + b_2 + \cdots + b_n$ is even.
* Natural to weight by $\frac{1}{\# \mathrm{Aut}}$.

FACTS (Norbury, 2009)

* $N_{g,n}$ = quasipolynomial in $b_1^2, b_2^2, \ldots, b_n^2$ of degree $3g-3+n$

i.e., $\exists$ polynomials $N_{g,n}^{(k)}$ s.t.

$$N_{g,n}(\underbrace{b_1, b_2, \ldots, b_k}_{\text{odd}}, \underbrace{b_{k+1}, b_{k+2}, \ldots, b_n}_{\text{even}}) = N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n)$$

* RECURSION

$N_{g,n}$ depends on $N_{g,n-1}$

$N_{g-1,n+1}$

$N_{g_1,n_1} \times N_{g_2,n_2}$ for $\begin{matrix} g_1 + g_2 = g \\ n_1 + n_2 = n+1 \end{matrix}$

* PARTIAL FORMULA

– If $|\alpha| = 3g-3+n$, then

$$[b_1^{2\alpha_1} b_2^{2\alpha_2} \cdots b_n^{2\alpha_n}] N_{g,n} = \frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}}{2^{5g-6+2n} \alpha_1! \alpha_2! \cdots \alpha_n!}$$

– $N_{g,n}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,n})$

* PARTIAL FORMULA + RECURSION = WITTEN'S CONJECTURE

EXAMPLES

$$N_{0,4}^{(0)} = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$$

$$N_{0,4}^{(2)} = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2)$$

$$N_{1,2}^{(0)} = \frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2 b_2^2 - 12b_1^2 - 12b_2^2 + 32)$$

$$N_{2,1}^{(0)} = \frac{1}{2^{16} \cdot 3^3 \cdot 5}(b_1^2 - 4)(b_1^2 - 16)(b_1^2 - 36)(5b_1^2 - 32)$$

Lemma: if $b_1 + b_2 + \cdots + b_n \leq 4g - 4 + 2n$, then $N_{g,n} = 0$.

EYNARD-ORANTIN INVARIANTS

INPUT: Riemann surface $C \subseteq \mathbb{C}^2$

OUTPUT: $\omega_{g,n}$ = multilinear form on C

RULE: $\omega_{g,n}$ depends on $\omega_{g,n-1}$

$\omega_{g-1,n+1}$

$\omega_{g_1,n_1} \times \omega_{g_2,n_2}$ for $\begin{matrix} g_1 + g_2 = g \\ n_1 + n_2 = n+1 \end{matrix}$

Theorem (Norbury, 2009):

The E-O invariants of $xy - y^2 = 1$ are

$$\omega_{g,n} = \sum_{b_1, b_2, \ldots, b_n = 1}^{\infty} b_1 b_2 \cdots b_n \, N_{g,n} \, z_1^{b_1 - 1} z_2^{b_2 - 1} \cdots z_n^{b_n - 1} \, dz_1 \, dz_2 \cdots dz_n$$

NEW IDEA

Goal: count lattice points in compactified moduli spaces

Example:

$\overline{\mathcal{M}}_{0,5} =$ 

1 copy 10 copies 15 copies

$$= \mathcal{M}_{0,5} \cup \underbrace{\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}}_{\text{10 copies}} \cup \underbrace{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}_{\text{15 copies}},$$

$$\overline{N}_{0,5} = N_{0,5}(b_1, b_2, b_3, b_4, b_5) + \sum_{\text{10 labellings}} N_{0,4}(b_i, b_j, b_k, 0) \, N_{0,3}(b_l, b_m, 0)$$

$$+ \sum_{\text{15 labellings}} N_{0,3}(b_i, b_j, 0) \, N_{0,3}(b_k, 0, 0) \, N_{0,3}(b_l, b_m, 0)$$

Hyperbolic viewpoint suggests nodes correspond to $b_i = 0$.

FACTS

* $N_{g,n}$ = quasipolynomial in $b_1^2, b_2^2, \ldots, b_n^2$ of degree $3g-3+n$

* PARTIAL FORMULA
  - $\overline{N}_{g,n} = N_{g,n}$ in top degree
  - $\overline{N}_{g,n}(0,0,\ldots,0) = \chi(\overline{\mathcal{M}}_{g,n})$

EXAMPLES

$$\overline{N}_{0,4}^{(0)} = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$$

$$\overline{N}_{0,4}^{(2)} = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 2)$$

$$\overline{N}_{1,2}^{(0)} = \frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2 b_2^2 + 36b_1^2 + 36b_2^2 + 192)$$

QUESTIONS

* Are coefficients of $\overline{N}_{g,n}$ always positive?
* What do they mean?
  Ideas: Hirzebruch - Riemann - Roch / toric varieties
* Does $\overline{N}_{g,n}$ arise from E-O invariants?
* $N_{g,n}$ counts factorisations in the symmetric group... what are the consequences?