Puzzle corner

Norman Do*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 12 is 1 July 2009. The solutions to Puzzle Corner 12 will appear in Puzzle Corner 14 in the September 2009 issue of the *Gazette*.

Soccer stats


Photo: Andrzej Skwarczyński

At some point during the soccer season, a player had scored in less than 80% of the matches. At some later point during the season, they had scored in more than 80% of the matches. Prove that there must have been some point when the player had scored in precisely 80% of the matches.

Loopy potatoes

You are given two potatoes. Show that it is possible to draw a non-trivial loop on each so that the two loops are congruent when considered as subsets of space.

Island tour

A circular island is divided into states by a number of chords of the circle. Consider a tour that starts and ends in the same state without passing through the

*Department of Mathematics and Statistics, The University of Melbourne, VIC 3010.
E-mail: N.Do@ms.unimelb.edu.au

intersection of any two borders. Prove that the tour must involve an even number of border crossings.

Table trouble

One thousand chairs, numbered from 1 to 1000, are equally spaced around a circular table. There is a plate on the table in front of each chair and these are also numbered from 1 to 1000. Is it always possible to rotate the table so that no chair has a number which matches the number on the corresponding plate?

Height differences

Twenty boys and twenty girls are paired off to form twenty couples. The difference in height between the boy and girl in each couple is no more than 10 centimetres. The boys and girls are then rearranged, so that the tallest boy is paired with the tallest girl, the second-tallest boy with the second-tallest girl, and so on. Prove that the difference in height between the boy and girl in each couple is still no more than 10 centimetres.

Prisoner perplexity

One hundred prisoners will each have a random integer from 1 to 100 written on their forehead. They will then be taken to a room where they can see the other prisoners' numbers, but not their own. Afterwards, they will be individually asked to guess the number on their own forehead. If at least one prisoner supplies the correct answer, then they will all be released; otherwise, they will all be executed. Given that the prisoners are allowed to discuss their strategy beforehand, show that they can devise a scheme which guarantees their freedom.

Photo: Robin Davis

Weighing coins

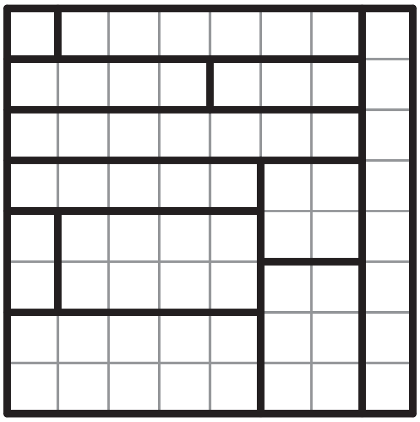
(1) You are given 68 coins, all of whose weights are distinct. Determine the lightest and heaviest coins by using a balance scale 100 times.
(2) You are given 61 coins. Two of them, whose weights are equal, are counterfeit and are either lighter or heavier than a genuine coin. Determine whether the counterfeit coins are lighter or heavier than a genuine coin by using a balance scale three times. (It is not necessary to identify the counterfeit coins.)
(3) You are given 12 coins. One of them is counterfeit and is either lighter or heavier than a genuine coin. Determine the counterfeit coin and whether it is lighter or heavier than a genuine coin by using a balance scale three times.

Solutions to Puzzle Corner 10

The \$50 book voucher for the best submission to Puzzle Corner 10 is awarded to James East.

Chopping a chessboard

Solution by James East: There are precisely 13 rectangles with integer side lengths and area at most 9: 1×1, 1×2, 1×3, 1×4, 2×2, 1×5, 1×6, 2×3, 1×7, 1×8, 2×4, 1×9 and 3×3. Therefore, 13 rectangles, no two of which are congruent, must tile an area greater than or equal to the sum of their areas, namely 72. So it is impossible to cut a regular 8×8 chessboard into 13 rectangles along its gridlines so that no two of the rectangles are congruent. However, the diagram shows that it is possible to cut it into 12 rectangles in such a manner.



How many numbers?

Solution by Simon Tyler: Suppose that the real numbers are $x_1 \leq x_2 \leq \ldots \leq x_n$. The problem states that $x_1 + x_2 + x_3 = 5$, $x_{n-2} + x_{n-1} + x_n = 7$, and $x_1 + x_2 + \cdots + x_n = 20$. It follows that $3x_3 \geq 5$, $3x_{n-2} \leq 7$, and $x_4 + x_5 + \cdots + x_{n-3} = 8$. These combine to give us

$$(n-6)x_3 \leq 8 \leq (n-6)x_{n-2} \Rightarrow \frac{5}{3}(n-6) \leq 8 \leq \frac{7}{3}(n-6) \Rightarrow \frac{66}{7} \leq n \leq \frac{54}{5}.$$

The fact that n is an integer gives the result $n = 10$.

Penny in a corner

Solution by John Graham: The locus is the part of the sphere $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 2\}$ lying in the cube $\{(x,y,z) \in \mathbb{R}^3 \mid -1 \leq x, y, z \leq 1\}$. The crux of the solution is the following result, whose proof is an exercise in analytic geometry.

> Let $(u_1, u_2, u_3) \in \mathbb{R}^3$ be a unit vector. Consider the disk with centre (q_1, q_2, q_3) which lies in a plane perpendicular to the unit vector (u_1, u_2, u_3). Then the disk is tangent to the yz-plane if and only if $q_1^2 + u_1^2 = 1$. Similarly, the disk is tangent to the zx-plane if and only if $q_2^2 + u_2^2 = 1$ and to the xy-plane if and only if $q_3^2 + u_3^2 = 1$.

It remains to determine for which $(q_1, q_2, q_3) \in \mathbb{R}^3$ we can simultaneously solve

$$q_1^2 + u_1^2 = 1, \qquad q_2^2 + u_2^2 = 1, \qquad q_3^2 + u_3^2 = 1, \qquad \text{and} \qquad u_1^2 + u_2^2 + u_3^2 = 1.$$

Observe that $q_1^2 + q_2^2 + q_3^2 = (1 - u_1^2) + (1 - u_2^2) + (1 - u_3^2) = 3 - u_1^2 - u_2^2 - u_3^2 = 2$. Hence, the centre of the disk must lie on the sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 2\}$. Also, $q_1^2 = 1 - u_1^2 \leq 1$ implies that $-1 \leq q_1 \leq 1$ and, similarly, we have $-1 \leq q_2 \leq 1$ and $-1 \leq q_3 \leq 1$. Hence, the centre of the disk must also lie in the cube $\{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x, y, z \leq 1\}$. Conversely, given (q_1, q_2, q_3) on the prescribed locus, we can find $u_1 = \pm\sqrt{1 - q_1^2}$, $u_2 = \pm\sqrt{1 - q_2^2}$, $u_3 = \pm\sqrt{1 - q_3^2}$, so these two necessary conditions are also sufficient.

Complete the table

Solution by Peter Nickolas: Let A_n, B_n, C_n and D_n denote the top-left, top-right, bottom-left and bottom-right quarters of the $2^{n+1} \times 2^{n+1}$ block of numbers in the top-left corner of the table. We will prove by induction that

- each row and column of A_n contains all of the numbers $0, 1, 2, \ldots, 2^n - 1$;
- the blocks A_n and D_n are identical; and
- the blocks B_n and C_n are identical and obtained by adding 2^n to each entry in A_n.

It can be checked that the statements hold for small values of n. In order to proceed by induction, we now suppose that they hold for some particular value of n, and consider the block A_{n+1}. Note that this is simply the amalgamation of the blocks A_n, B_n, C_n and D_n. By the inductive hypothesis, each of its rows and columns contain all of the numbers $0, 1, 2, \ldots, 2^{n+1} - 1$. The occurrence of $0, 1, 2, \ldots, 2^{n+1} - 1$ in B_{n+1} is then ruled out, but there are no other constraints. So the pattern of entries in B_{n+1} will follow the pattern of entries in A_{n+1}, but with each entry larger by 2^{n+1}. This reasoning applies similarly to C_{n+1}. Each column above and each row to the left of D_{n+1} contains all of the numbers $2^{n+1}, 2^{n+1} + 1, \ldots, 2^{n+2} - 1$, so the pattern of entries in D_{n+1} will follow the pattern for A_{n+1}, and the induction is complete.

This result allows the calculation of any entry of the table to be reduced to the calculation of an entry which is closer to the top-left corner. Let us index the entries of the table as $a_{i,j}$, for $i, j = 0, 1, 2, \ldots$. Then we have

$$a_{455,122} = a_{199,122} + 256 = a_{71,122} + 128 + 256 = a_{7,58} + 128 + 256 = \cdots,$$

and continuing in a similar manner, we find that $a_{455,122} = 445$.

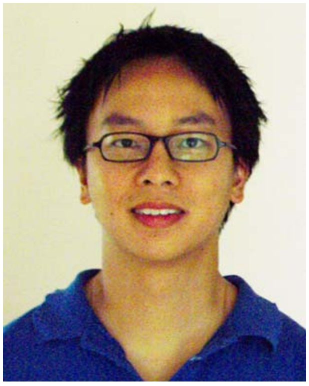
It also follows that the value of $a_{i,j}$ can be obtained by adding all the powers of 2 that correspond to the places in which the binary expansions of i and j differ. For example, $455_{10} = 111000111_2$ and $122_{10} = 1111010_2$, so $a_{455,122} = 256 + 128 + 32 + 16 + 8 + 4 + 1 = 445$.

The way to Heaven

Solution by Dave Johnson:

(1) Ask one of the people, ‘Would your companion say that the left path is the way to Heaven?’ If the answer is YES, take the right path; if the answer is NO, take the left path.

(2) Ask one of the people, 'If I asked your companion whether the left path is the way to Heaven, would the answer be GEB?'. If the answer is GEB, take the right path; if the answer is MIU, take the left path.
(3) Ask person A, 'Is person B more likely to tell the truth than person C?' If the answer is YES, you deduce that either A or B is the random answerer; if the answer is NO, then you deduce that either A or C is the random answerer. In either case, you identify a person who is not the random answerer. You may now ask that person, 'If I asked you whether the left path is the way to Heaven, would the answer be YES?' If the answer is YES, take the left path; if the answer is NO, take the right path.



Norman is a PhD student in the Department of Mathematics and Statistics at The University of Melbourne. His research is in geometry and topology, with a particular emphasis on the study of moduli spaces of algebraic curves.