Puzzle corner

Norman Do*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 10 is 1 January 2009. The solutions to Puzzle Corner 10 will appear in Puzzle Corner 12 in the May 2009 issue of the *Gazette*.

Chopping a chessboard

A regular 8×8 chessboard is cut into n rectangles along its gridlines so that no two of the rectangles are congruent. What is the maximum possible value of n?

Photo: S. Schleicher

How many numbers?

Some real numbers are written on a blackboard. It is known that the sum of the numbers is 20, the smallest sum using three of the numbers is 5 and the largest sum using three of the numbers is 7. How many numbers are there on the blackboard?

Penny in a corner

A disk of radius 1 moves so that its circumference is continually in contact with the xy-plane, the yz-plane, and the zx-plane in three-dimensional space. Find the locus of the centre of the disk.

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Complete the table

Below is part of a table with infinitely many rows and infinitely many columns. The entry in the top left corner is 0 and every other square is labelled with the smallest non-negative integer which does not appear in that row to the left of it or in that column above it.

0	1	2	3	4	5	
1	0	3	2	5	4	
2	3	0	1	6	7	
3	2	1	0	7	6	
4	5	6	7	0	1	
5	4	7	6	1	0	

By completing the table, or otherwise, determine which number occurs in the 123rd column and the 456th row.

The way to Heaven

You arrive at a fork in the road. One path leads to the village named Heaven while the other leads to the village named Hell, but you do not know which is which. Consider the following three scenarios.

Photo: L. Emerson

(1) Standing before you are two people. One always tells the truth and one always lies, but you do not know which is which. How can you discover the way to Heaven by asking only one yes-or-no question to one of the two?

(2) Standing before you are two people. One always tells the truth and one always lies, but you do not know which is which. They both understand English but respond using their native language. Fortunately, you remember that the words for YES and NO in their native language are GEB and MIU, but you seem to have forgotten which is which. How can you discover the way to Heaven by asking only one yes-or-no question to one of the two?

(3) Standing before you are three people. One always tells the truth, one always lies and one answers yes-or-no questions randomly, but you do not know which is which. How can you discover the way to Heaven by asking only two yes-or-no questions, each directed to just one of the three?

Solutions to Puzzle Corner 8

The \$50 book voucher for the best submission to Puzzle Corner 8 is awarded to Gerry Myerson.

Watchful wombats

Solution by Ivan Guo: We will prove the statement by induction on n, the number of wombats. The base case of $n = 1$ is trivial since a lone wombat cannot be watched. Suppose that the statement is true for $n = k$ and consider now the case $n = k+2$. Suppose that the wombats A and B have the smallest distance between them and note that they must be watching each other. Now remove A and B from the group. If any of the remaining wombats were watching A or B, let them now turn to watch the closest other wombat in the remaining group. In this new group of size k, each wombat is once again watching its closest neighbour. So by the inductive hypothesis, one of them — say X — is not being watched. Now rewind back to the days when A and B were still around. Wombat X could not have been watched by anyone from the group of size k. Furthermore, X was definitely not watched by A or B, who were too busy watching each other. Therefore, we have proven the statement by induction for all odd positive integers n.

Rubik's cube

Solution by James McCoy: Let the numbers on the faces of the cube be a, b, c, d, e, f, where a is opposite b, c is opposite d, and e is opposite f. Then the products assigned to the vertices are ace, acf, ade, adf, bce, bcf, bde, bdf and Rubik has noticed that

$$ace + acf + ade + adf + bce + bcf + bde + bdf = 2008.$$

Conveniently, this equation can be equivalently expressed in the factorised form

$$(a + b)(c + d)(e + f) = 2008.$$

Now $2008 = 2^3 \times 251$ and each of $a+b$, $c+d$, $e+f$ is at least 2. So the possibilities for the factors $a+b$, $c+d$, $e+f$, up to ordering, are $2, 4, 251$ or $2, 2, 502$. Therefore, the sum of the numbers can be $2 + 4 + 251 = 257$ or $2 + 2 + 502 = 506$.

Crowded subsets

Solution by Iwantme Voucher[!]: A crowded set $A \subseteq S$ is uniquely determined by its common difference d and its smallest element a. By extending the set $\{a, a + d, a + 2d, \ldots\}$ as far as possible, one obtains a crowded set if and only if $a \leq d$ and $a + d \leq n$. Thus, it suffices to count pairs of positive integers (a, d) with $a \leq d \leq n - a$. For a fixed value of a, there are $n - 2a + 1$ such pairs if $2a \leq n$, and none otherwise. Therefore, the total number of crowded subsets of S is

$$\sum_{a=1}^{\lfloor n/2 \rfloor} n - 2a + 1 = \begin{cases} 1 + 3 + 5 + \cdots + (n-1) = n^2/4 & \text{if } n \text{ is even,} \\ 2 + 4 + 6 + \cdots + (n-1) = (n^2-1)/4 & \text{if } n \text{ is odd.} \end{cases}$$

Clock confusion

Solution by Paul Emanuel: Let the angle of the hour hand past noon be α and let the angle of the minute hand past the hour be β. Furthermore, let $\alpha = \alpha_0 + (k\pi)/6$ and $\beta = \beta_0 + (\ell\pi)/6$, where $\alpha_0, \beta_0 \in [0, \pi/6)$ and $k, l \in \{0, 1, 2, \ldots, 11\}$. Then a time is indistinguishable if the proportion of the hour hand past some hour equals the proportion of the minute hand through the hour, and the proportion of the minute hand past some hour equals the proportion of the hour hand through the hour. In other words, we require the following two equations to be true.

$$\alpha_0 \div \frac{\pi}{6} = \frac{6\alpha}{\pi} - k = \frac{\beta}{2\pi}$$
$$\beta_0 \div \frac{\pi}{6} = \frac{6\beta}{\pi} - \ell = \frac{\alpha}{2\pi}.$$

Solving these equations simultaneously leads to

$$\alpha = \frac{2\pi}{143}(12k + \ell) \quad \text{and} \quad \beta = \frac{2\pi}{143}(k + 12\ell).$$

When $k = \ell$, we obtain $\alpha = \beta$ and the hands are overlapping, in which case the time is not indistinguishable. For all other $k, \ell \in \{0, 1, 2, \ldots, 11\}$, α and β are not multiples of $\pi/6$, and so for any such fixed pair (k, ℓ) there is exactly one indistinguishable time. Therefore, there are $12 \times 12 - 12 = 132$ such occasions.

Alternative solution based on work submitted by Sam Krass: Consider a third hand which moves 12 times as fast as the minute hand. The ambiguous times correspond precisely to when the hour hand and third hand coincide. In the 12-hour period between noon and midnight, the third hand travels around the clock 144 times while the hour hand travels around the clock once. Therefore, they coincide precisely 143 times, but 11 of these are when the hour and minute hands are overlapping. Therefore, the number times which are indistinguishable is $143 - 11 = 132$.

Give and Take

Solution by David Angell: Give can choose 16 handfuls of 6 coins and 1 handful of 4 coins. Since there are 17 handfuls altogether, the game finishes as soon as one player has received 9 handfuls, comprising 52 or 54 coins. The other player receives the remaining 48 or 46 coins, so Give can be sure of getting at least 46 coins.

Alternatively, Take can adopt the strategy of accepting any handful of 6 or more coins and refusing any handful of 5 or fewer coins. If the game finishes with Take accepting 9 handfuls, then Take receives at least 54 coins and Give receives at most 46. However, if the game finishes otherwise, then Give receives at most 9 handfuls of at most 5 coins — at most 45 coins in total. Therefore, we conclude that Give can be sure of getting at least 46 coins, but not more.

Playing with polynomials

Solution by Gerry Myerson:

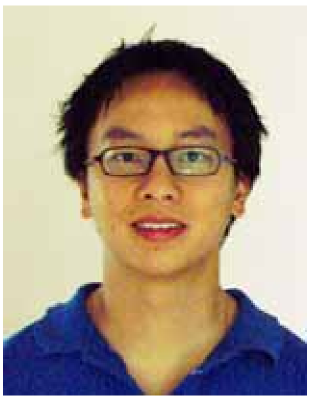
(1) The value of the quadratic polynomial at $x = -1$ is initially -98 and finally 100. Furthermore, each change to its coefficients altered the value at $x = -1$ by ± 1. Therefore, at some point in time the value of the polynomial at $x = -1$ must have been 0. So there was a point in time when the polynomial had -1 as a root. So, by virtue of being a monic quadratic polynomial with integer coefficients, there was a point in time when the polynomial had integer roots.

(2) In theory, one question suffices. Since π is transcendental, there cannot be two distinct polynomials with integer coefficients that take on the same value at π. Thus, $P(x)$ is uniquely determined by the value of $P(\pi)$.
In a somewhat more practical sense, two questions suffice. Ask for $P(1)$ and suppose that the answer is m. Then ask for $P(b)$ where b is an integer greater than m. Then the coefficients of $P(x)$ can be read off from the base-b representation of $P(b)$. This is true because $P(1) < b$ implies that every coefficient of P is less than b, and base b representation of a positive integer is unique.

(3) Take the polynomial

$$F(x, y) = \frac{(x+y)(x+y+1)}{2} + x.$$

Consider the sequence of triangular numbers $0, 1, 3, 6, 10, \ldots$ given by $T_m = (m(m+1))/2$ for $m = 0, 1, 2, \ldots$. Then for every non-negative integer n there is a unique non-negative integer s such that $T_s \leq n < T_{s+1}$. Since $T_{s+1} - T_s = s + 1$, it follows that every non-negative integer can be written uniquely as $n = T_s + r$ with $0 \leq r \leq s$. Now $T_s + r = F(r, s - r)$, and the one-to-one correspondence is clear.



Norman is a PhD student in the Department of Mathematics and Statistics at The University of Melbourne. His research is in geometry and topology, with a particular emphasis on the study of moduli spaces of algebraic curves.