Puzzle corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 6 is 1 May 2008. The solutions to Puzzle Corner 6 will appear in Puzzle Corner 8 in the July 2008 issue of the *Gazette*.

Pasting pyramids

Take one solid pyramid with a square base, where all edges have unit length. Take another solid pyramid, this time with a triangular base, where all edges have unit length. Paste the pyramids together by matching two of the triangular faces. How many faces does the resulting solid have?

Million dollar question

In a particular nation, the currency consists of notes of four different values: $1, $10, $100, and $1000. Can one have exactly half a million notes with a total value of exactly one million dollars?

Photo: Nauris Mozolevs

Collecting coins

One hundred coins of various denominations lie in a row on a table. Alex and Bree alternately take a coin from either end of the row. Alex goes first which means that Bree will take the last coin on the table. Show that Alex can guarantee to end up with at least as much money as Bree.

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Symmetric sets

Determine all finite sets S of points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

Double deck

A deck of 50 cards contains two cards labelled 1, two cards labelled 2, two cards labelled 3, and so on. There are 25 people seated around a circular table, with each holding two cards from the deck. Every minute, each person passes the card that they are holding with the smaller label to the right. Prove that someone must eventually have two cards labelled with the same number.

Photo: Ian Barber

Monks on a mountain

Photo: ©Miro Schaap

(1) A monk begins to ascend a mountain at dawn on Monday, reaching the summit at dusk on the same day. After spending the night at the peak, he begins to descend the mountain at dawn on Tuesday, reaching the bottom by dusk on the same day. Prove that at some precise time of day, the monk was at exactly the same altitude on Monday as he was on Tuesday.

(2) Two monks start at sea level at points A and B on opposite sides of a mountain chain. There is a path running from A to B which has finitely many peaks and troughs and never dips below sea level. Is it always possible for the monks to travel along the whole path from one end to the other so that they always remain at the same altitude?

Solutions to Puzzle Corner 4

The \$50 book voucher for the best submission to Puzzle Corner 4 is awarded to Peter Pleasants.

Digital sequences

Solution by Gerry Myerson: If the leftmost digit is k, then the next digit greater than k in the sequence must be $k+1$, the next digit greater than k after that must be $k+2$, and so on. Therefore, the digits greater than k must occur in the order $k+1, k+2, \ldots, 9$ and similarly, the digits less than k must occur in the order $k-1, k-2, \ldots, 0$. Since the digits less than k may appear in any of the 9 positions after the leftmost, there are $\binom{9}{k}$ ways to write such a sequence where the leftmost digit is k. So the total number of ways is

$$\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \cdots + \binom{9}{9} = 2^9 = 512.$$

Blindfold balance

Solution by James East: Divide the coins into two groups, one with 13 coins and the other with 17 coins. If there are n coins showing heads in the first group, then there must be $17 - n$ coins showing heads in the second group. Now simply turn over all the coins in the second group and there will be n coins showing heads in both groups.

Chicken O'Nuggets

Solution by Kevin McAvaney: The smallest positive number of nuggets we can order is six. Also, note that if there are no colossal packs, then the number of nuggets is a multiple of three. We can get $6k$ nuggets by ordering an appropriate number of large packs and $6k + 3$ nuggets by ordering one humongous pack and an appropriate number of large packs, where k is a positive integer. To order 43 nuggets, one must use 0, 1 or 2 colossal packs which leaves either 43, 23 or 3 nuggets to make up with large and humongous packs, respectively. Clearly, none of these is possible, so one cannot order 43 nuggets.

However, the number of nuggets can be $38 = 1 \times 20 + 3 \times 6$, $39 = 13 \times 3$ or $40 = 2 \times 20$. Hence, any number larger than 43 can be ordered by adding the appropriate multiple of three.

Photo: Hans Thoursie

Plane passengers

Solution based on work submitted by Ben Smith: When the last passenger boards the plane, the seat remaining either belongs to them or to the first passenger. Note that there is no preference shown by the remaining passengers toward one or the other of those two seats. Therefore, the probability that the last passenger gets their own seat is simply 50%.

Crime investigation

Solution by Natalie Aisbett: Denote the judge's original plan involving no more than 91 questions by P. If the witness may answer at most one question falsely, then the judge may modify the scheme as follows. They should divide the 91 questions of P into 13 groups consisting of $13, 12, 11, \ldots, 3, 2, 1$ questions, respectively. (Note that $1 + 2 + 3 + \cdots + 13 = 91$.) At the end of each group, the judge should ask the following additional check question: 'Did you lie while answering any of the questions in the previous group?'

If the answer to the check question is ever 'No', then there were no wrong answers in the previous group. At the end of questioning, if there were no 'Yes' answers to the check questions, then all questions of P were answered truthfully. In this case, the judge uses $91 + 13 = 104$ questions.

If the judge receives the answer 'Yes' to the check question after the kth group, the judge should repeat this group and then continue to ask the remaining questions of P without using the check questions. In this case, the judge needs k check questions and $14 - k$ repetitions of the questions of the kth group. In all, the judge requires $91 + k + (14 - k) = 105$ questions.

A facetious function

Solution by Peter Pleasants: The effect of the function f on a positive integer n is to reverse the binary representation of n. For example, $f(100) = 19$ since the binary representations of 100 and 19 are 1100100_2 and 10011_2, which are the reverse of each other. (We ignore any leading zeros after the reversal is performed.) We will continue to use a subscript 2 to denote binary representations, S to denote an arbitrary string of binary digits, and $\overline{S}$ to denote the reversal of the binary string S.

We start by observing that $\overline{1}_2 = 1_2$, $\overline{11}_2 = 11_2$ and $\overline{S0}_2 = \overline{S}_2$. These correspond precisely to the first three defining equations for the function f. We also have

$$\overline{S01}_2 = 10\overline{S}_2 = 2 \times \overline{S1}_2 - \overline{S}_2$$

and

$$\overline{S11}_2 = 11\overline{S}_2 = 3 \times \overline{S1}_2 - 2 \times \overline{S}_2.$$

These correspond to the last two defining equations of f and confirm that f does indeed have the effect of reversing the binary digits of its argument.

The number of integers $1 \leq n \leq 1988$ with $f(n) = n$ is therefore equal to the number of such n whose binary representations are palindromic. To calculate this, we notice that there are $2^{\lceil \ell/2 \rceil - 1}$ numbers with symmetric binary representations of length ℓ. Hence, there are

$$1 + 1 + 2 + 2 + 4 + 4 + 8 + 8 + 16 + 16 + 32 = 94$$

such numbers up to 2047, the largest 11-digit binary number. From this we have to subtract the number of positive integers with palindromic binary representations in the range 1989 to 2047. The binary representation of 1988 is 11111000100 and the only numbers greater than this with 11 binary digits and palindromic binary representations are $2015 = 11111011111_2$ and $2047 = 11111111111_2$. Hence there are 92 positive integers less than or equal to 1988 for which $f(n) = n$.

A mathematician is lost in the woods...

Solution by Ian Wanless:

(1) The mathematician should walk along the circumference of a circle of area A square kilometres. Such a path has length $2\sqrt{\pi A}$ kilometres.

(2) The mathematician should walk along a semicircular arc of radius $\sqrt{2A/\pi}$ kilometres. Such a semicircle has area A square kilometres and such a path has length $\sqrt{2\pi A}$ kilometres.

(3) A circle of radius $\sqrt{A/\pi}$ kilometres has area A square kilometres so a circle of any greater radius necessarily contains a point not in the woods. So by walking $\sqrt{A/\pi}$ kilometres, it must be possible to reach the boundary of the woods.

(4) Suppose that the woods lies on the coordinate plane, where one unit represents one kilometre. The mathematician should walk in a straight line from $(0,0)$ to $(\frac{1}{\sqrt{3}},1)$ and then in another straight line to the point $(\frac{\sqrt{3}}{2},\frac{1}{2})$. At this point in time, the mathematician is distance 1 from the origin. He should continue walking around the unit circle until he reaches the point $(-1,0)$. He should then complete his path by walking in a straight line from $(-1,0)$ to $(-1,1)$. Such a path has length

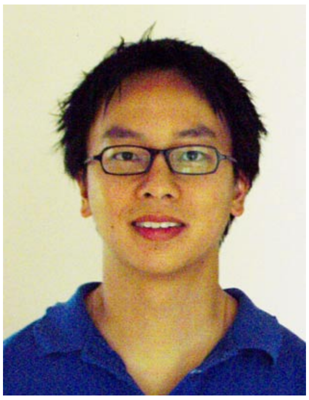
$$\sqrt{3}+\frac{7\pi}{6}+1=6.397\ldots \text{ kilometres.}$$

The convex hull of the path includes the circle of radius 1 around the mathematician's starting point, so it must take him to the edge of the forest at some point.

(5) Suppose that the woods lies on the coordinate plane, where one unit represents one kilometre. Starting at the point $(-\frac{1}{2},0)$, the mathematician should walk clockwise along the circle with radius 1 and centre $(\frac{1}{2},0)$ for a distance of $\arctan(2)-\arctan(\frac{1}{2})$. At this point, the mathematician is facing the point $(0,1)$ and distance $\frac{1}{2}$ from it. He should continue by walking there in a straight line. The remainder of the path is the same, but reflected through the line $x=0$. Therefore, the mathematician should finish at the point $(\frac{1}{2},0)$. Such a path has length

$$2\arctan(2)-2\arctan\left(\tfrac{1}{2}\right)+1=2.287\ldots \text{ kilometres.}$$

The mathematician's path cannot be placed completely within a strip one kilometre wide.



Norman is a PhD student in the Department of Mathematics and Statistics at The University of Melbourne. His research is in geometry and topology, with a particular emphasis on the study of moduli spaces of algebraic curves.