# Puzzle corner 5

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 5 is 1 March 2007. The solutions to Puzzle Corner 5 will appear in Puzzle Corner 7 in the May 2008 issue of the *Gazette*.

### Pricey pills

A sick patient has been advised to take exactly one pill of Xenitec and exactly one pill of Yenitec every day. The patient has a full bottle of each type of pill. One day, he pours one Xenitec pill onto a table and accidentally pours two Yenitec pills onto the table as well. Unfortunately, the three pills look identical and the patient has no way of telling them apart. Furthermore, since the pills are extremely expensive, the patient would rather not have to discard any of them. How can the patient save all three pills, but still maintain a proper daily dosage?

### Marching band

The members of a marching band are arranged in a rectangular array. In order to aid visibility, the leader of the band decides to rearrange each column so that the heights are non-decreasing from front to back. He then decides to rearrange each row so that the heights are non-decreasing from left to right. Prove that the columns are still in non-decreasing order of height from front to back.

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### Silly soldiers

A row of freshly recruited soldiers is facing a sergeant who gives them the command to turn right. Some of the soldiers turn right, while the rest of them turn left. After one second of confusion, they all try to correct their mistake in the following way. If a soldier sees the face of their neighbour, they assume they have turned the wrong way and decide to turn around; otherwise, the soldier remains facing the same way. If any two soldiers are facing each other, then there is another second of confusion before they try to correct their mistake in the same way. This continues until the situation stabilises and no two soldiers are facing each other. If there are $n$ soldiers, then what is the maximum length of time that it will take for the situation to stabilise?

### The lazy fly

A rectangular room is 30 feet long, 12 feet in height, and 12 feet wide. A lazy fly is resting in the middle of a wall at the end of the room, at a point one foot down from the ceiling. In the middle of the opposite wall, one foot up from the floor is a tiny scrap of food. The fly is too lazy to fly and decides to walk to the food at the other end of the room. What is the minimum distance that the fly must walk?

### Dinner party handshakes

My wife and I were invited to a dinner party attended by nine other couples, making a total of 20 people. A certain amount of handshaking took place subject to two conditions: no one shook his or her own hand and no couple shook hands with each other. Afterwards, I became curious and asked everybody else at the party how many people they shook hands with. Given that I received nineteen different answers, how many people could my wife have shaken hands with?

### Counting the digits

You are given three infinite lists of numbers: $A$, $B$, and $C$. List $A$ contains all numbers of the form $10^k$, where $k$ is a positive integer, written in base 10. Lists $B$ and $C$ contain the same numbers written in base 2 and base 5, respectively.

| $A$ | $B$ | $C$ |
|---|---|---|
| 10 | 1010 | 20 |
| 100 | 1100100 | 400 |
| 1000 | 1111101000 | 13000 |
| $\vdots$ | $\vdots$ | $\vdots$ |

Prove that for every integer $n > 1$, there is exactly one number appearing in lists $B$ or $C$ which has exactly $n$ digits.

### Tennis anyone?

(1) Suppose that 1000 tennis players want to play in a tennis tournament. In each round, they are paired up and the winner progresses to the next round. If the number of players in a round is odd, then one player is chosen to automatically progress to the next round before the pairing occurs. This continues until the final round which contains only one match and the winner of this match is crowned the champion. What is the total number of matches played in the whole tournament?

(2) You have been granted magical powers which have enabled you to reach the Wimbledon tennis final. However, you have been warned that your powers will run out during the match, after which your opponent will have a distinct advantage. What score do you want it to be when your powers disappear in order to maximize your chances of hanging on for a win?

(3) Alex and Bobbi are about to play a tennis match where the winner is the first player to reach twelve games. It has been decided that Alex will serve first but they are considering using one of the two following serving schemes: the alternating serves scheme in which the two players take turns to serve or the winner serves scheme in which the winner of a game serves in the next. It is known that Alex has 0.71 chance of winning his serve while Bobbi has only 0.67 chance of winning her serve. Which scheme should Alex choose to maximize his chances of winning the match?

## Solutions to Puzzle Corner 3

The \$50 book voucher for the best submission to Puzzle Corner 3 is awarded to Codrut Grosu.

### Milk and tea

*Solution by Mike Hirschhorn:* Note that the two cups contain the same amount of liquid before and after the mixing. Therefore, the amount of milk that has been displaced is equal to the amount of tea that has been displaced. It follows that the percentage of milk in the tea is equal to the percentage of tea in the milk.

### Lockers

*Solution by Gordon Clarke:* The state of locker number $n$ is toggled by student $d$ if and only if $d$ is a factor of $n$. Therefore, the final state of locker number $n$ is open if and only if $n$ has an odd number of factors. However, $d$ is a factor of $n$ if and only if $\frac{n}{d}$ is a factor of $n$, so the factors come in pairs unless $d = \frac{n}{d}$, in which case $n = d^2$.

Therefore, only the square numbers have an odd number of factors, so the lockers which remain open are numbered 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100.

### Irrational punch

*Solution by David Angell:* Of course, one application of the hole punch is not enough. It is also clear that two punches will not suffice, since there are infinitely many points on the perpendicular bisector of the centres which have the same rational distance from each.

Now we will show that three punches centred at the points $(0, 0)$, $(\alpha, 0)$ and $(0, \alpha)$ suffice, for some choice of the real number $\alpha$. Note that the point $(x, y)$ survives all three punches if and only if

$$x^2 + y^2 = r^2, \quad (x - \alpha)^2 + y^2 = s^2, \quad \text{and} \quad x^2 + (y - \alpha)^2 = t^2\,,$$

for some rational numbers $r$, $s$, and $t$.

However, it follows from the first and second equations that $x = (\alpha^2 + r^2 - s^2)/2\alpha$ and similarly, from the first and third equations, that $y = (\alpha^2 + r^2 - t^2)/2\alpha$. Now substituting these expressions for $x$ and $y$ into the first equation and clearing fractions yields

$$(\alpha^2 + r^2 - s^2)^2 + (\alpha^2 + r^2 - t^2)^2 = 4r^2\alpha^2\,.$$

This implies that $\alpha$ satisfies a degree four polynomial with rational coefficients. Therefore, by choosing a value of $\alpha$ which is not a root of a degree four polynomial with rational coefficients (for example, $\alpha = \pi$) we can ensure that the three punches dispose of the entire plane.

### Integral averages

*Solution by Peter Pleasants:* If $S \subseteq \{1, 2, \ldots, n\}$ has integral average $a$ then the set $n+1-S$ obtained by reflection in the midpoint $(n + 1)/2$ has integral average $n+1-a$. Note that $n+1-S = S$ if and only if $S$ is symmetric about $(n + 1)/2$. Hence, the parity of $T_n$ is the same as the parity of $S_n$, the number of non-empty subsets of $\{1, 2, 3, \ldots, n\}$ with integral average that are symmetric about $(n + 1)/2$.

The average of any set symmetric about $(n + 1)/2$ must of course be $(n + 1)/2$. When $n$ is even, this is not an integer, so $S_n = 0$ and $T_n$ is also even.

When $n$ is odd, $(n + 1)/2$ is an integer and the subsets of $\{1, 2, \ldots, n\}$ that are symmetric about $(n + 1)/2$ and have integral average come in pairs, where the sets of each pair differ only in the fact that one contains $(n + 1)/2$ while the other does not. Once we discard the empty set, this shows that $S_n$, and hence also $T_n$, is odd. Therefore, in each case the parity of $T_n$ is the same as the parity of $n$.

### Silver matrices

*Solution based on work submitted by Sam Krass:*

(a) For each value of $k$, let us call the union of the $k$th row and the $k$th column a cross. Note that each off diagonal entry of the matrix belongs to exactly two crosses while each diagonal entry belongs to exactly one cross. Now suppose that one of the numbers occurs off the diagonal $a$ times and on the diagonal

$b$ times. Then since each number occurs in each cross exactly once, we have

$$2a + b = n.$$

Therefore, if $n$ is odd, then $b \geq 1$ and every number from 1 to $2n-1$ must lie on the diagonal. So unless $n = 1$, there will be more numbers than diagonal entries and no silver matrix can exist. In particular, there is no silver matrix for $n = 2007$.

(b) We will show that whenever a $k \times k$ silver matrix exists, then a $2k \times 2k$ silver matrix also exists. Consider the matrix

$$X = \left[\begin{array}{c|c} A & B \\ \hline C & A \end{array}\right],$$

where $A$ is a $k \times k$ silver matrix, $B$ is a $k \times k$ matrix such that every row and every column contains the numbers $\{2k, 2k+1, \ldots, 3k-1\}$ in some order, and $C$ is a $k \times k$ matrix such that every row and every column contains the numbers $\{3k, 3k+1, \ldots, 4k-1\}$ in some order. For example, we can construct $B$ and $C$ by placing the entries in the first row in numerical order and then cyclically shifting each subsequent row.

By construction, every cross of $X$ contains a cross of $A$, a row of $B$ and a column of $C$ or a cross of $A$, a column of $B$ and a row of $C$. So each cross contains $\{1, 2, \ldots, 2k-1\} \cup \{2k, 2k+1, \ldots, 3k-1\} \cup \{3k, 3k+1, \ldots, 4k-1\}$ as desired.

Now since a $1 \times 1$ silver matrix is easy to construct, it follows that silver matrices exist whenever $n$ is a power of 2.

### Puzzles for prisoners

*Solution based on work submitted by Codrut Grosu*

(1) Of course, no strategy is guaranteed to save all the prisoners, since there is no hope for the unlucky prisoner at the back of the line. However, we will show that the remaining 99 prisoners can be saved!

Note that each prisoner need only know their number modulo 100 in order to be freed. The prisoner at the back of the line calls out the sum of the remaining 99 numbers, modulo 100. (If the sum is 0 modulo 100, then the prisoner should call out the number 100.) The next prisoner can then calculate the modulo 100 sum of the 98 numbers before them and subtract this from the total sum, thereby calculating their own number. Subsequent prisoners will know the total sum of the 99 people to be saved, can see all of the numbers in front of them, and has heard the numbers of everyone behind them. Therefore, they can calculate their own number.

(2) Let us suppose, for ease of exposition, that the one hundred prisoners have names which are simply the integers from 1 to 100. Furthermore, we may assume that the boxes are also numbered from 1 to 100. Suppose that box $k$ contains the number $P(k)$, so that $P$ is simply a permutation of the numbers from 1 to 100.

The optimal strategy is actually quite simple! Each prisoner starts by opening the box with his or her number on it. They then open the box matching the number they found in the previous box. They continue in this way until they reach a box with their own name inside or have opened 50 boxes.

But why on earth does this strategy work? Since $P$ is a permutation of the numbers from 1 to 100, we can write it as a union of disjoint cycles in the standard way. Now we simply note that every prisoner will find his or her name if and only if there is no cycle in $P$ of length greater than 50. So all that is left for us to do is calculate the probability that a permutation of 100 elements contains no cycle of length greater than 50.

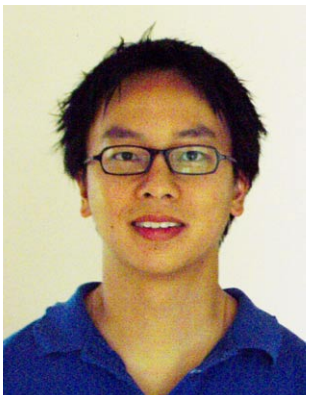
Let us begin by calculating the probability that a permutation on 100 elements contains a cycle of length $k > 50$. Well, there are $\binom{100}{k}$ ways to choose the elements of the cycle, $(k-1)!$ ways to put those elements in cyclic order, and $(100-k)!$ ways to permute the remaining elements. The product of these is $100!/k$ and since there are $100!$ permutations, the desired probability is simply $\frac{1}{k}$. Since it is impossible to have more than one cycle of length greater than 50, the probability of having no cycle of length greater than 50 is given by the expression

$$1 - \left(\frac{1}{51} + \frac{1}{52} + \cdots + \frac{1}{100}\right).$$

However, the reader is invited to show that for all positive integers $n$,

$$1 - \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) > 1 - \ln 2 \approx 0.307\,.$$

Therefore, using this strategy, the prisoners will be freed with a probability that exceeds thirty per cent!



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