Puzzle corner 4

Norman Do*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to `N.Do@ms.unimelb.edu.au` or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 4 is 1 November 2007. The solutions to Puzzle Corner 4 will appear in Puzzle Corner 6 in the March 2008 issue of the *Gazette*.

I would like to thank Jamie Simpson of Curtin University for sending me some interesting problems, one of which appears as the first entry in this Puzzle Corner. If you have your own favourite problems which you would like to share, then I would most certainly welcome your contribution. Problems may be submitted by the same means as solutions, as described above.

Digital sequences

The digits from 0 to 9 are written in a row such that each digit other than the leftmost is within one of some digit to the left of it. In how many ways can this be done?

Blindfold balance

Thirty coins lie on a table with precisely seventeen of them showing heads. Your task, should you choose to accept it, is to form two groups of coins, not necessarily of the same size, such that each contains the same number of coins showing heads. Unfortunately, you happen to be blindfolded and cannot feel the difference between the two sides of a coin. How can you perform the task?

*Department of Mathematics and Statistics, The University of Melbourne, VIC 3010.
E-mail: N.Do@ms.unimelb.edu.au

Chicken O'Nuggets

A certain Irish fast food chain sells Chicken O'Nuggets in large, humongous or colossal packs. A large pack contains 6 O'Nuggets, a humongous pack contains 9 O'Nuggets, and a colossal pack contains 20 O'Nuggets. What is the largest number of Chicken O'Nuggets that is impossible to order?

Plane passengers

There are one hundred people waiting to board a plane with one hundred seats. However, the first passenger to board has forgotten their seat number and decides to occupy a random seat. Each subsequent passenger takes their assigned seat if it is available and otherwise takes a random unoccupied seat. What is the probability that the last passenger to board finds their seat unoccupied?

Crime investigation

While questioning a witness, a judge is only allowed to ask questions which are to be answered 'yes' or 'no'. The judge has carefully calculated that, as long as the witness answers every question truthfully, then she can solve the case in no more than 91 questions. One or more of these questions may depend on the answers to previous questions. Show that the judge can solve the case in no more than 105 questions if it is known that the witness may answer at most one question falsely.

A facetious function

The following question appeared in the 1988 International Mathematical Olympiad which was held in Canberra.

A function f defined on the positive integers and taking positive integer values is given by

$$\begin{aligned} f(1) &= 1, \\ f(3) &= 3, \\ f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n), \end{aligned}$$

for all positive integers n. Determine with proof the number of positive integers less than or equal to 1988 for which $f(n) = n$.

A mathematician is lost in the woods...

(1) A mathematician is lost in the woods. He knows that its area is A square kilometres and that it has no holes. Show that he can escape by walking no more than $2\sqrt{\pi A}$ kilometres.

(2) A mathematician is lost in the woods. He knows that its area is A square kilometres and that it is convex in shape. Show that he can escape by walking no more than $\sqrt{2\pi A}$ kilometres.
(3) A mathematician is lost in the woods. He knows that its area is A square kilometres and that it is convex in shape. Show that if he is allowed to consult a person who knows the shortest way out, then he can escape by walking no more than $\sqrt{A/\pi}$ kilometres.
(4) A mathematician is lost in the woods. He knows that it is in the shape of a half plane and that he is precisely one kilometre away from the edge of the woods. Show that he can escape by walking no more than 6.4 kilometres.
(5) A mathematician is lost in the woods. He knows that it is in the shape of a strip one kilometre wide and infinite in length. Show that he can escape by walking no more than 2.3 kilometres.

Solutions to Puzzle Corner 2

The \$50 book voucher for the best submission to Puzzle Corner 2 is awarded to Natalie Aisbett.

Coffee and doughnuts

Solution by Jeremy Ottenstein: Let C and D be the price in dollars for a coffee and a doughnut, respectively. Let B and G be the number of boys and girls, respectively. From the information given in the problem, we have the equation

$$BC + GD = BD + GC + 1 \Rightarrow (B - G)(C - D) = 1.$$

This implies that $B-G$ is a divisor of 1 so that $B-G = \pm 1$. Since there are more girls than boys in the class, it follows that $G = B+1$. Apart from the obvious fact that B and G must also be nonnegative integers, this is all that can be determined about B and G.

Solitaire

Solution by Christopher Chen: Let us say that the game can be solved if it is possible to end with one piece remaining on the board. We conjecture that the game can be solved if and only if n is not a multiple of 3.

Part 1: The game cannot be solved when n is a multiple of 3.

Start by labelling each square of the board with the sum of its coordinates taken modulo 3. Each square will be given a label 0, 1 or 2 with the property that any three consecutive squares have distinct labels. Let X_0, X_1 and X_2 denote the number of pieces on squares labelled by 0, 1 and 2, respectively. If n is a multiple of 3, it is easy to see that at the beginning of the game, $X_0 = X_1 = X_2$.

Observe that whenever a move is made, we lose two pieces on squares of different labels, but gain a piece on a square of the third label. Therefore, the triple ($X_0 \mod 2, X_1 \mod 2, X_2 \mod 2$) is invariant under any move. Note that our initial value is $(0, 0, 0)$ or $(1, 1, 1)$ while our desired final value should be $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$. Therefore, it is impossible to end with only one piece remaining on the board when n is a multiple of 3.

Part 2: The game can be solved when n is not a multiple of 3.

If $n \equiv 1 \pmod 3$, then it is possible to remove the pieces on the perimeter of the original square, leaving us with an $(n-2) \times (n-2)$ block. We note that $n - 2 \equiv 2 \pmod 3$.

If $n \equiv 2 \pmod 3$, then it is possible to remove the pieces on the perimeter as well as those pieces which are adjacent to those on the perimeter, leaving us with an $(n-4) \times (n-4)$ block. We note that $n - 4 \equiv 1 \pmod 3$.

Therefore, by alternately applying the above two procedures, we will be left with either the $n = 1$ case or the $n = 2$ case, both of which can obviously be solved.

Glass half full

Solution by Gerry Myerson: The key is to use the symmetry of the glass — no extra equipment is required! Simply tilt the glass as far as you can without spilling any water. If the bottom of the glass is not covered entirely by water, then the glass is less than half full. If the bottom is covered with water to spare, then the glass is more than half full. And, of course, if the bottom is covered with no water to spare, then the glass is exactly half full.

Secret salaries

Solution by Peter Pleasants: The solution involves six pieces of paper. Alice writes the same random number on two pieces of paper and passes one to Bob. On a third piece of paper, Bob adds his salary to the number he received from Alice and passes the result to Clare. On a fourth piece of paper, Clare adds her salary to the number she received from Bob and passes the result to Alice. On a fifth piece of paper, Alice adds her salary to the number she received from Clare and passes the result to Bob. On the sixth and final piece of paper, Bob adds his salary to the second number he received from Alice and passes the result to Clare.

Let R be Alice's random number and A, B, and C be Alice, Bob, and Clare's salaries, respectively. Then after the whole procedure, each person has two pieces of paper with the following numbers.

- Alice: R and $R + B + C$
- Bob: R and $R + A + B + C$
- Clare: $R + B$ and $R + A + 2B + C$

Now each person need only take the difference between the two numbers that they have. Alice will obtain the sum of Bob and Clare's salaries from which she can easily determine the sum of all three salaries and, hence, the average. Bob and Clare each obtain the sum of all three salaries and may similarly determine the average. Furthermore, no one can extract any more information about any individual salary.

Ambulatory ants

Solution by by Sam Krass:

(1) Let n be the number of ants and T be the time that an ant takes to complete a lap of the circular path. To an observer who cannot distinguish between the ants, it would appear that when a collision occurs, the ants simply pass by each other, rather than changing directions. Therefore, at time T, the positions of the ants are identical to their starting positions, perhaps up to some permutation.

However, note that the ants always maintain the same cyclic ordering around the circular path. So at time T, each ant has been cyclically shifted along k places from their initial positions for some $0 \leq k < n$. Therefore, at time nT, every ant will be in its starting location.

(2) *As noted by several astute readers, the problem is not true unless the ants all walk at the speed of one metre per minute. Your humble author wishes to apologise for this omission in the original problem statement.*

(a) Once again, to an observer who cannot distinguish between the ants, it would appear that when a collision occurs, the ants simply pass by each other, rather than changing directions. Therefore, after two minutes, the positions of the ants are identical to their starting positions, perhaps up to some permutation. However, note that the ants always maintain the same ordering along the stick. Therefore, after two minutes, every ant will be in its starting location.

(b) Using our earlier observations, we note that if an ant begins at distance x metres from the left end of the stick, then after one minute, there must be an ant at distance $1 - x$ metres from the left end of the stick. Since the ants always maintain the same ordering along the stick, every ant is in its starting location after one minute if and only if the initial locations of the ants are symmetric about the centre of the stick.

(3) Let the ants which begin on the second hand, minute hand and hour hand of the clock be S, M and H, respectively. We will show that after twelve hours, S has travelled the greatest number of times around the clock.

The crucial observation is that at no time during the twelve hours does one ant overtake another ant. This is due to the fact that when a faster hand overtakes a slower hand, the ant which was on the faster hand ends up on the slower hand. It is clear that at 1 second after noon, S has travelled further than M, who has travelled further than H. Therefore, at every moment during the twelve hours, S has travelled at least as far as M, who has travelled at least as far as H.

Let the number of revolutions completed by S, M and H be R_S, R_M and R_H, respectively. From our earlier observations, it follows that $R_H \leq R_M \leq R_S$. Furthermore, since S cannot overtake H, we also have the inequality $R_S \leq R_H + 1$. However, the three ants together have travelled as far as the three hands of the clock together. Since the second hand performs 720 revolutions, the minute hand performs 12 revolutions and the hour hand performs 1 revolution, we obtain $R_H + R_M + R_S = 720 + 12 + 1 = 733$. The only integral solution for (R_H, R_M, R_S) which satisfies the above inequalities is $(R_H, R_M, R_S) = (244, 244, 245)$. Therefore, the ant which begins on the second hand has travelled the greatest number of times around.

An unusual identity

Solution by Natalie Aisbett: Arrange the numbers in two rows as follows.

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{array}$$

Suppose that there is a value of k such that $n < a_k < b_k$. Then it is clear that the numbers

$$\begin{array}{ccccccc} & & & a_k & a_{k+1} & \cdots & a_n \\ b_1 & b_2 & \cdots & b_k & & & \end{array}$$

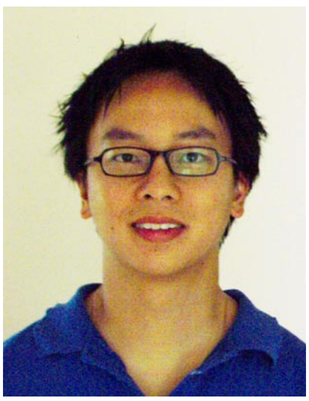
are all at least as large as a_k. Hence, these $n+1$ numbers must belong to the set $\{n+1, n+2, \ldots 2n\}$, which contradicts the fact that they are distinct. Using analogous arguments, one can show that there is no value of k such that one of the following occurs:

$$n < a_k < b_k, \qquad n < b_k < a_k, \qquad n \geq a_k > b_k, \quad \text{or} \quad n \geq b_k > a_k.$$

In other words, we have proven that for every value of k, the larger of a_k and b_k is in the set $\{n+1, n+2, \ldots, 2n\}$ while the smaller of a_k and b_k is in the set $\{1, 2, \ldots, n\}$.

Therefore, we have the following chain of equalities:

$$\begin{aligned} &|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| \\ &\quad = [\max(a_1, b_1) - \min(a_1, b_1)] + \cdots + [\max(a_n, b_n) - \min(a_n, b_n)] \\ &\quad = [\max(a_1, b_1) + \cdots + \max(a_n, b_n)] - [\min(a_1, b_1) + \cdots + \min(a_n, b_n)] \\ &\quad = [(n+1) + (n+2) + \cdots + 2n] - [1 + 2 + \cdots + n] \\ &\quad = [(n+1) - 1] + [(n+2) - 2] + \cdots + [2n - n] \\ &\quad = n + n + \cdots + n \\ &\quad = n^2. \end{aligned}$$



Norman is a PhD student in the Department of Mathematics and Statistics at The University of Melbourne. His research is in geometry and topology, with a particular emphasis on the study of moduli spaces of algebraic curves.