PROBABILITY MEASURES AND COMPLEX ANALYSIS 29/04/11
Week 8 – Norman Do

Young diagrams, Plancherel measure, and the semicircle distribution

Reference: Kerov's "Transition probabilities for continual Young diagrams..."

Motivation 1: Plancherel measure $\overset{?}{\longleftrightarrow}$ semicircle distribution

Motivation 2: Recall the Stieltjes transform $S_\mu(z) = \int \frac{\mu(x)\,dx}{z-x}$.

$$\begin{tikzcd}
\text{bounded Jacobi matrix } J \arrow[r] & \text{continued fraction } \Phi(z) \arrow[r] \arrow[d] & \text{measure } \mu \text{ with } S_\mu(z)=\Phi(z) \\
\text{orthogonal polynomials } B_n \text{ where } \phi_n(z)=\frac{A_n(z)}{B_n(z)} & \text{convergents } \phi_n(z) \arrow[l] \arrow[r] & \text{measure } \mu_n \text{ with } S_{\mu_N}(z)=\phi_n(z) \arrow[u, "\text{converge in weak topology}"']
\end{tikzcd}$$

Fact: $\deg A_n = n-1$, $\deg B_n = n$ and roots of A_n and B_n "interlace"

Young diagrams:

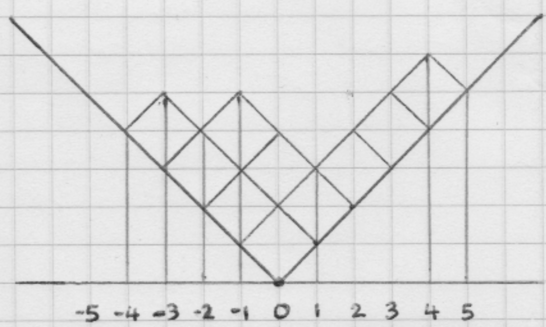
① parts $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$

② downward steps $h_1 > h_2 > h_3 > \cdots$ $\quad (h_i = \lambda_i - i + \frac{1}{2})$

③ peaks / valleys $x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n < x_{n+1}$

$\Sigma x_i = \Sigma y_i$

Example:



① 5, 2, 2, 1, 0, 0, 0, ...

② $4\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -2\frac{1}{2}, -4\frac{1}{2}, -5\frac{1}{2}, -6\frac{1}{2}, \ldots$

③ peaks : $-3, -1, 4$
valleys : $-4, -2, 1, 5$

Associate to λ the "transition measure" μ with

$$S_\mu(z) = \frac{(z-y_1)\cdots(z-y_n)}{(z-x_1)\cdots(z-x_n)(z-x_{n+1})} = \sum_{i=1}^{n+1} \frac{\mu_i}{z-x_i}. \quad \text{(partial fraction expansion)}$$

So $\mu = \sum_{i=1}^{n+1} \mu_i \delta_{x_i}$.

Fact: • $\mu_1, \mu_2, \dots, \mu_{n+1} > 0 \iff$ x's and y's interlace

• $\sum \mu_i = 1$

Proof: • Use $\mu_m = \dfrac{(x_m - y_1) \cdots (x_m - y_n)}{(x_m - x_1) \cdots \cancel{(x_m - x_m)} \cdots (x_m - x_{n+1})}$

• Use $\sum \mu_i = \sum \mathrm{Res}_{x_i} S_\mu(z) = -\mathrm{Res}_\infty S_\mu(z)$

Rectangular diagrams: Fix an interval $[a,b]$. A rectangular diagram is a piecewise linear $\lambda: \mathbb{R} \to \mathbb{R}$ with

• $\lambda'(x) = \pm 1$ except on a finite set,

• $\lambda(x) = |x - c|$ for $x \notin [a,b]$.

Fact:

$$D_0 = \left\{ \begin{matrix} \text{rectangular diagrams} \\ \text{supported on } [a,b] \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{measures with finite} \\ \text{support in } [a,b] \end{matrix} \right\} = \mathcal{M}_0$$

$$\begin{tikzcd} \text{rectangular diagram } \lambda \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\text{peaks/valleys}"'] \arrow[dr, "?"] & \text{discrete measure } \mu = \sum \mu_i \delta_{x_i} \arrow[d, leftrightarrow, "S\text{-transform}"] \\ \dfrac{\prod (z - y_i)}{\prod (z - x_i)} \arrow[r, equal, "\text{partial fraction expansion}"'] & S_\mu(z) = \sum \dfrac{\mu_i}{z - x_i} \end{tikzcd}$$

Fact: Let $\hat{\lambda}(x) = \dfrac{\lambda(x) - |x - c|}{2}$. Then $S_\mu(z) = \dfrac{1}{z - c} \exp\left[S_{\hat{\lambda}}(z) \right]$

Proof: Use $\int f'(t)\, d\hat{\lambda}(t) = f(0) + \sum f(y_i) - \sum f(x_i)$ with $f(t) = \ln(z - t)$.

Diagrams: (Limits of rectangular diagrams)

A diagram is a continuous $\lambda: \mathbb{R} \to \mathbb{R}$ with

• $|\lambda(x) - \lambda(y)| \le |x - y|$

• $\lambda(x) = |x - c|$ for $x \notin [a,b]$.

Fact:

$$\begin{array}{ccccc} & & D_0 & \xleftrightarrow{1:1} & \mathcal{M}_0 & & \\ & & \cap\!\!| & & \cap\!\!| & & \\ \text{(uniform convergence)} & & D & \xleftrightarrow{\text{homeomorphic}} & \mathcal{M} & & \text{(weak convergence)} \\ & & \| & & \| & & \\ & & \left\{\begin{matrix}\text{diagrams supported} \\ \text{on } [a,b]\end{matrix}\right\} & & \left\{\begin{matrix}\text{measures supported} \\ \text{on } [a,b]\end{matrix}\right\} & & \end{array}$$

Proof: Use $S_\mu(z) = \frac{1}{z-c} \exp\left[S_{\hat{\lambda}}(z)\right]$ and the fact that $D_0 \subseteq D$ and $\mathcal{M}_0 \subseteq \mathcal{M}$ are dense.

Plancherel measure: If $|\lambda| = N$ is a partition, define $\rho(\lambda) = \frac{(\dim \lambda)^2}{N!}$.

The measure is normalised since $\sum_{|\lambda|=N} (\dim \lambda)^2 = N!$.

Hook length formula: $\dim \lambda$ = # ways to "grow" the Young diagram of λ

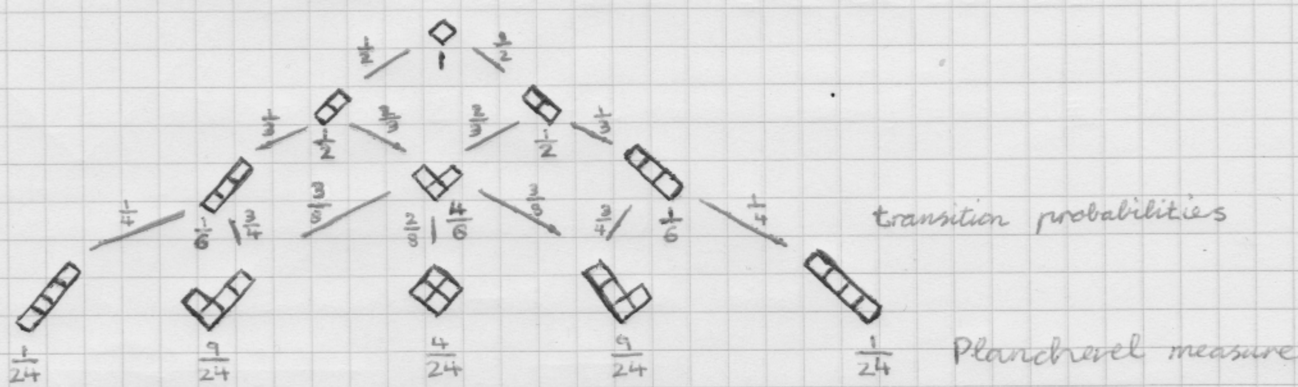
$$= \frac{N!}{\pi \text{ hook lengths}}$$

Example:

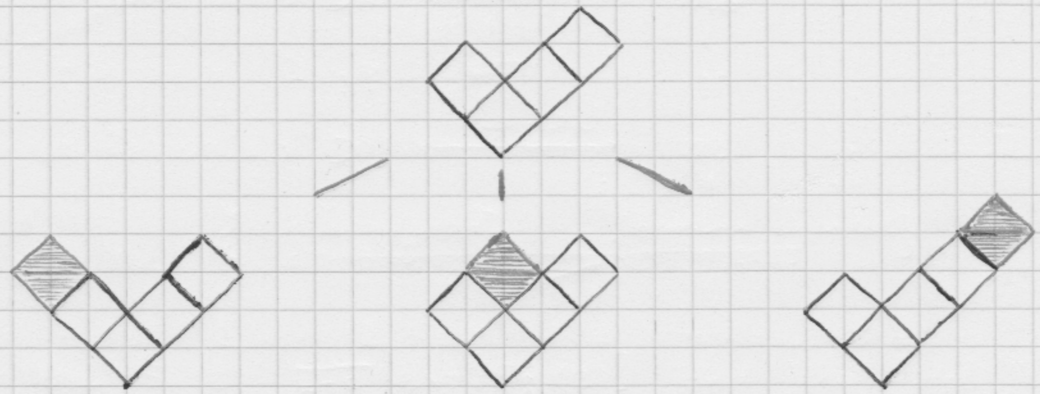
8	6	3	2	1
4	2			
3	1			
1				

$$\dim (5,2,2,1) = \frac{10!}{8 \cdot 6 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 2 \cdot 3 \cdot 1 \cdot 1} = 525$$

Markov chain on Young's lattice:



Question: Given λ, what are the transition probabilities?



Answer: They come from the transition measure!

Proof: • Use $p(\lambda \to \lambda + \square) = \frac{\dim(\lambda+\square)}{(N+1)\dim\lambda}$.

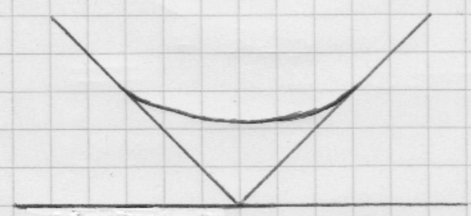
• Use hook length formula to write $p(\lambda \to \lambda + \square) = \frac{\prod hl(\lambda)}{\prod hl(\lambda+\square)}$.

• The product "collapses" to

$$p(\lambda \to \lambda + \square) = \frac{x_m - y_1}{x_m - x_1} \cdots \frac{x_m - y_{m-1}}{x_m - x_{m-1}} \cdot \frac{x_m - y_m}{x_m - x_{m+1}} \cdots \frac{x_m - y_n}{x_m - x_{n+1}} = \mu_m.$$

Limit shape theorem: Let
$$\Lambda(x) = \frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4-x^2}\right), \quad \text{for } |x| \leq 2$$
$$= |x|, \quad \text{for } |x| > 2.$$

For all $\varepsilon > 0$, $\lim_{N\to\infty} P\left\{ |\lambda| = N \;\middle|\; \begin{array}{l}\lambda \text{ normalised to area } 1 \\ \text{is } \varepsilon\text{-close to } \Lambda\end{array} \right\} = 1.$



Corollary: Transition measure for limit shape is $\frac{1}{2\pi}\sqrt{4-x^2}\,dx$.

Interesting facts:

- Plancherel measure converges to a Gaussian random process.
- The limit shape displays Gaussian fluctuations.
- The limit shape appears unexpectedly in the root separation of orthogonal polynomials.