LATTICE POINTS IN MODULI SPACES OF CURVES

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There appear to be only two essentially distinct ways to understand intersection numbers on moduli spaces of curves — via Hurwitz numbers or symplectic volumes. In this talk, we will consider polynomials defined by Norbury which bridge the gap between these two pictures. They appear in the enumeration of lattice points in moduli spaces of curves and it appears that their coefficients store interesting information. We will also describe a connection between these polynomials and the topological recursion defined by Eynard and Orantin.

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Moduli spaces of curves

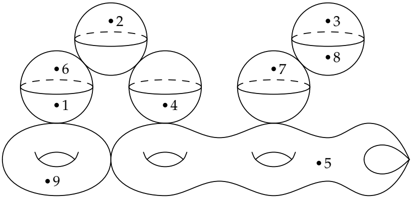
- Moduli spaces of curves

$$\mathcal{M}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ smooth algebraic curves with distinct} \\ \text{points labelled from 1 up to } n \end{array} \right\}$$

- Deligne–Mumford compactification

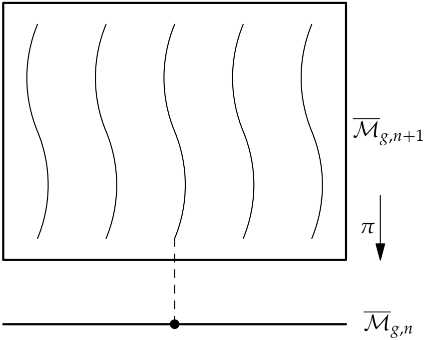
$$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{l} \text{genus } g \text{ stable algebraic curves with distinct} \\ \text{smooth points labelled from 1 up to } n \end{array} \right\}$$

A stable curve may be nodal but its components satisfy $2g - 2 + n > 0$.



- The complex dimension of $\overline{\mathcal{M}}_{g,n}$ is $3g - 3 + n$.

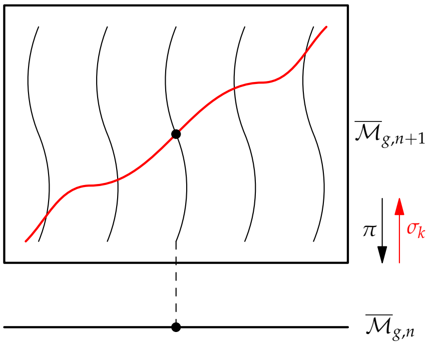
Intersection theory on $\overline{\mathcal{M}}_{g,n}$



$\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ forgets the point labelled $n+1$

the fibre over a point in $\overline{\mathcal{M}}_{g,n}$ is the curve associated to that point

Intersection theory on $\overline{\mathcal{M}}_{g,n}$



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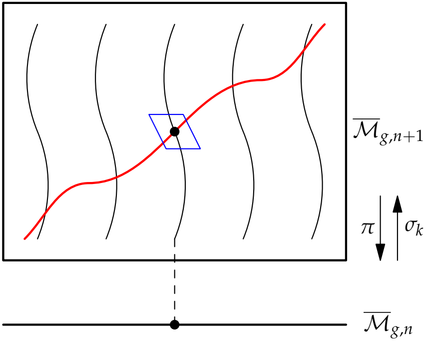

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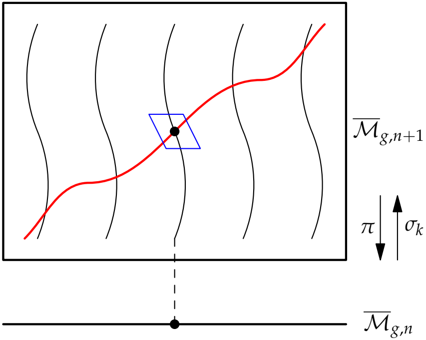

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- Define $\psi_k = c_1[\sigma_k^* \mathcal{L}] \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ for $k = 1, 2, \ldots, n$.
- For $|\boldsymbol{\alpha}| = 3g - 3 + n$, Witten considers the psi-class intersection number

$$\langle \tau_{\alpha_1} \tau_{\alpha_2} \cdots \tau_{\alpha_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n} \in \mathbb{Q}.$$

Witten's conjecture

If $F(t_0, t_1, t_2, \ldots) = \sum_{\mathbf{d}} \prod_{k=0}^{\infty} \frac{t_k^{d_k}}{d_k!} \langle \tau_0^{d_0} \tau_1^{d_1} \tau_2^{d_2} \cdots \rangle$, then $\frac{\partial^2 F}{\partial t_0^2}$ satisfies KdV.

- In two-dimensional quantum gravity, we want to integrate over the space of metrics on a surface.

  LINEARISATION
  $\longrightarrow$ tilings of surfaces
  $\longrightarrow$ matrix integrals
  $\longrightarrow$ KdV hierarchy

  LOCALISATION
  $\longrightarrow$ integration over conformal classes

- There are now several proofs of Witten's conjecture.

HURWITZ NUMBERS	VOLUMES OF MODULI SPACES
Okounkov and Pandharipande	Kontsevich
Kazarian and Lando	Mirzakhani
Kim and Liu	

Kontsevich's proof

- A ribbon graph of type (g, n) is a cell decomposition of a genus g surface with n faces labelled from 1 up to n.
- A metric ribbon graph is a ribbon graph with a positive real number assigned to every edge — the metric endows each face with a perimeter.
- For fixed positive real numbers $b_1, b_2, \ldots, b_n$,

$$\mathcal{M}_{g,n} \cong \left\{ \begin{array}{l} \text{metric ribbon graphs of type } (g, n) \text{ with} \\ \text{perimeters of lengths } b_1, b_2, \ldots, b_n \end{array} \right\}.$$

Kontsevich's combinatorial formula

$$\sum_{|\boldsymbol{\alpha}|=3g-3+n} \langle \tau_{\alpha_1} \tau_{\alpha_2} \cdots \tau_{\alpha_n} \rangle \prod_{k=1}^{n} \frac{(2\alpha_k - 1)!!}{s_k^{2\alpha_k+1}} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|\mathrm{Aut}\,\Gamma|} \prod_{e \in E(\Gamma)} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

$$\boxed{\text{intersection theory on } \overline{\mathcal{M}}_{g,n} \longleftrightarrow \text{enumeration of ribbon graphs}}$$

Other proofs

- Okounkov and Pandharipande

$$H_{g,\boldsymbol{\mu}} = \#\left\{ \begin{array}{l} \text{genus } g \text{ branched covers of } \mathbb{P}^1 \text{ with branching profile} \\ \boldsymbol{\mu} \text{ over } \infty \text{ and simple branching at } r \text{ other points} \end{array} \right\}$$

$$= \frac{r!}{|\mathrm{Aut}\,\boldsymbol{\mu}|} \prod_{k=1}^{n} \frac{\mu_k^{\mu_k}}{\mu_k!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)\cdots(1-\mu_n\psi_n)}$$

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- Mirzakhani

$$V_{g,n}(\mathbf{L}) = \mathrm{VOL} \left\{ \begin{array}{l} \text{genus } g \text{ hyperbolic surfaces with } n \text{ geodesic} \\ \text{boundaries of lengths } L_1, L_2, \ldots, L_n \end{array} \right\}$$

$$= \sum_{|\boldsymbol{\alpha}|+m=3g-3+n} \frac{(2\pi^2)^m \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\psi_2^{\alpha_2}\cdots\psi_n^{\alpha_n}\kappa_1^m}{2^{|\boldsymbol{\alpha}|}\alpha_1!\alpha_2!\cdots\alpha_n!m!} L_1^{2\alpha_1} L_2^{2\alpha_2} \cdots L_n^{2\alpha_n}.$$

$$\boxed{\text{intersection theory on } \overline{\mathcal{M}}_{g,n} \longleftrightarrow \text{volumes of moduli spaces}}$$

- These volume polynomials satisfy a recursion of the form

$$V_{g,n} = \text{a certain integration over } V_{g-1,n+1},\ V_{g,n-1} \text{ and } V_{g_1,n_1} \times V_{g_2,n_2}$$

for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$.

Idea

Count lattice points in moduli spaces of curves.

- Let $N_{g,n}(b_1, b_2, \ldots, b_n)$ be the number of metric ribbon graphs of type (g, n) with integer edge lengths and perimeters equal to $b_1, b_2, \ldots, b_n$, weighted by $\frac{1}{\#\text{ automorphisms}}$.
- This gives a discrete approximation to the volume of the moduli space and a combinatorial problem similar to the Hurwitz problem.
- Since $N_{g,n}(b_1, b_2, \ldots, b_n)$ counts tilings of surfaces, it can be calculated using a matrix integral.

Theorem

These lattice point enumerations satisfy a recursion of the form

$$N_{g,n} = \textit{a certain summation over } N_{g-1,n+1}, N_{g,n-1} \textit{ and } N_{g_1,n_1} \times N_{g_2,n_2}$$

for $g_1 + g_2 = g$ *and* $n_1 + n_2 = n + 1$.

Examples of lattice point polynomials

Corollary

The lattice point enumeration $N_{g,n}(b_1, b_2, \ldots, b_n)$ is a degree $3g-3+n$ quasi-polynomial in $b_1^2, b_2^2, \ldots, b_n^2$.

g	n	k	$N_{g,n}^{(k)}(b_1, b_2, \ldots, b_n)$
0	3	0 or 2	1
1	1	0	$\frac{1}{48}(b_1^2 - 4)$
0	4	0 or 4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2)$
1	2	0	$\frac{1}{384}(b_1^2 + b_2^2 - 4)(b_1^2 + b_2^2 - 8)$
1	2	2	$\frac{1}{384}(b_1^2 + b_2^2 - 2)(b_1^2 + b_2^2 - 10)$
2	1	0	$\frac{1}{2^{16}.3^3.5}(b_1^2 - 4)(b_1^2 - 16)(b_1^2 - 36)(5b_1^2 - 32)$
3	1	0	$\frac{1}{2^{25}.3^6.5^2.7}(5b_1^4 - 188b_1^2 + 1152)\prod_{k=1}^{5}(b_1^2 - 4k^2)$

Coefficients of lattice point polynomials

Theorem

- *The top degree part of $N_{g,n}(b_1, b_2, \ldots, b_n)$ stores all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.*
- *The quasi-polynomial $N_{g,n}$ satisfies $N_{g,n}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,n})$.*

Proof.

- The lattice point enumeration $N_{g,n}$ approximates the volume of the moduli space up to a constant factor. Kontsevich and Mirzakhani tell us that this volume stores all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- Consider the following meromorphic function and calculate its value at infinity in two ways.

$$\sum_{b_1, b_2, \ldots, b_n = 1}^{\infty} N_{g,n}(b_1, b_2, \ldots, b_n)\, z^{b_1 + b_2 + \cdots + b_n} \qquad \square$$

Eynard–Orantin topological recursion

- INPUT: A Riemann surface C with two meromorphic functions x and y.
- OUTPUT: Meromorphic multilinear forms $\omega_{g,n}(z_1, z_2, \ldots, z_n)$ on C.
- RULE: These multilinear forms satisfy a recursion of the form

  $$\omega_{g,n} = \text{a certain residue over } \omega_{g-1,n+1}, \omega_{g,n-1} \text{ and } \omega_{g_1,n_1} \times \omega_{g_2,n_2}$$

  for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$.

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for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$.

Theorem

The multilinear forms associated to the curve $xy - y^2 = 1$ are given by

$$\begin{aligned} \omega_{g,n}(z_1, z_2, \ldots, z_n) &= \sum_{b_1, b_2, \ldots, b_n = 1}^{\infty} N_{g,n}(b_1, b_2, \ldots, b_n) \prod_{k=1}^{n} b_k z_k^{b_k - 1} dz_k \\ \omega_{g,0} &= \chi(\mathcal{M}_{g,0}) \end{aligned}$$

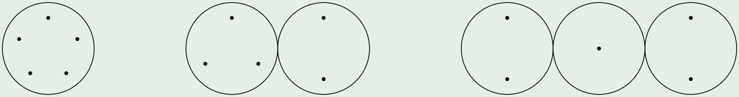
Lattice points in compactified moduli spaces of curves

Idea

Count lattice points in compactified moduli space of curves

Example

Points in $\overline{\mathcal{M}}_{0,5}$ represent curves of the following types.



$\mathcal{M}_{0,5}$ — 1 labelling

$\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}$ — 10 labellings

$\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$ — 15 labellings

$$\begin{aligned}\overline{N}_{0,5}(b_1,b_2,b_3,b_4,b_5) &= N_{0,5}(b_1,b_2,b_3,b_4,b_5) \\ &+ \sum_{\text{10 terms}} N_{0,4}(b_i,b_j,b_k,0)\cdot N_{0,3}(b_\ell,b_m,0) \\ &+ \sum_{\text{15 terms}} N_{0,3}(b_i,b_j,0)\cdot N_{0,3}(b_k,0,0)\cdot N_{0,3}(b_\ell,b_m,0)\end{aligned}$$

Compactified lattice point polynomials

Fact

- The lattice point enumeration $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is a degree $3g - 3 + n$ quasi-polynomial in $b_1^2, b_2^2, \ldots, b_n^2$.
- The quasi-polynomials $N_{g,n}$ and $\overline{N}_{g,n}$ agree to leading order — so the top degree part of $\overline{N}_{g,n}$ stores all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- The quasi-polynomial $\overline{N}_{g,n}$ satisfies $\overline{N}_{g,n}(0, 0, \ldots, 0) = \chi(\overline{\mathcal{M}}_{g,n})$.

g	n	k	$\overline{N}^{(k)}_{g,n}(b_1, b_2, \ldots, b_n)$
0	3	0 or 2	1
1	1	0	$\frac{1}{48}(b_1^2 + 20)$
0	4	0 or 4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 2)$
1	2	0	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2b_2^2 + 48b_1^2 + 48b_2^2 + 192)$
1	2	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2b_2^2 + 48b_1^2 + 48b_2^2 + 84)$

Questions

Claim

The compactified lattice point enumeration $\overline{N}_{g,n}$ seems to be the right thing to study (as opposed to $N_{g,n}$).

- *Are the coefficients of $\overline{N}_{g,n}$ always positive?*
  We conjecture (and hope) that the answer is "yes".
- *What geometric information is stored in the coefficients of $\overline{N}_{g,n}$?*
  The quasi-polynomials $\overline{N}_{g,n}$ have a Hirzebruch–Riemann–Roch flavour.
  So perhaps the coefficients store dimensions of spaces of sections.
- *Do the quasi-polynomials $\overline{N}_{g,n}$ arise from the Eynard–Orantin topological recursion?*
  We conjecture (and hope) that the answer is "yes".
- *There are relations between $N_{g,n}$ and matrix integrals, factorisations in the symmetric group, integrable hierarchies, etc. What are the consequences of these connections?*