COUNTING LATTICE POINTS IN COMPACTIFIED MODULI SPACES OF CURVES

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21 July 2011 @ Park City Mathematics Institute

Simple combinatorial objects called ribbon graphs can be interpreted as lattice points in the moduli space of curves. The enumeration of ribbon graphs produces polynomials whose top degree coefficients are intersection numbers and whose constant term is the Euler characteristic of the moduli space. On the other hand, the intermediate coefficients remain a complete mystery. In this talk, we'll see how to calculate these polynomials and highlight connections with Gromov-Witten theory, completed cycles, and topological recursion.

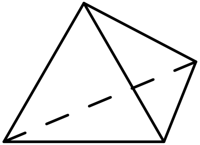
TILING A SURFACE WITH POLYGONS

In how many ways can you obtain a genus g surface by gluing together the edges of a given set of polygons?

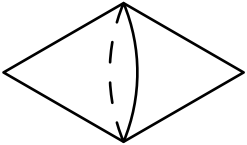
- Let the polygons be numbered $1, 2, \ldots, n$ and have $b_1, b_2, \ldots, b_n$ edges.
- The edges of the polygons form a graph on the resulting surface called a ribbon graph of type (g, n).
- We won't allow two adjacent edges to be glued together — in other words, we won't allow degree one vertices in the ribbon graph.
- Denote the enumeration by $N_{g,n}(b_1, b_2, \ldots, b_n)$, where we attach the weight $\frac{1}{\#\text{Aut }\Gamma}$ to a ribbon graph Γ.

Example

You should be able to calculate that $N_{0,4}(3, 3, 3, 3) = 8$.



2 labellings



6 labellings

RIEMANN SURFACES AND RIBBON GRAPHS

Theorem [Strebel, 1984]

INPUT: A Riemann surface C, distinct points $p_1, p_2, \ldots, p_n$ on C, and positive real numbers $r_1, r_2, \ldots, r_n$.
OUTPUT: The unique quadratic differential on C whose closed horizontal trajectories fill C, having double poles at $p_1, p_2, \ldots, p_n$ with residues $r_1, r_2, \ldots, r_n$.
BYPRODUCT: A ribbon graph of type (g, n) formed from the non-closed horizontal trajectories, with a positive length attached to every edge.

Corollary

The space $\mathcal{M}_{g,n}$ is homeomorphic to the space of ribbon graphs where

- every vertex has degree at least three;
- a positive length is attached to every edge; and
- the perimeter of face k is r_k.

Idea

Interpret ribbon graphs with integer edge lengths as lattice points in the moduli space of curves.

A RECURSION FOR RIBBON GRAPHS

Theorem [Norbury, 2010]

The following recursion can be used to effectively compute $N_{g,n}(b_1, b_2, \ldots, b_n)$.

$$(b_1 + b_2 + \cdots + b_n)\, N_{g,n}(\mathbf{b}_S) = \sum_{\substack{\{i,j\} \subseteq S \\ p+q=b_i+b_j}} pq\, N_{g,n-1}(p, \mathbf{b}_{S\setminus\{i,j\}})$$

$$+ \sum_{\substack{i \in S \\ p+q+r=b_i}} pqr \left[N_{g-1,n+1}(p, q, \mathbf{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}} N_{g_1,|I|+1}(p, \mathbf{b}_I) N_{g_2,|J|+1}(q, \mathbf{b}_J) \right]$$

Note that $N_{g,n}$ relies on $N_{g-1,n+1}$, $N_{g,n-1}$, and $N_{g_1,n_1} \times N_{g_2,n_2}$ for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$.

Proof.

Think about what happens when you remove an edge from the graph. □

Corollary

The enumeration $N_{g,n}(b_1, b_2, \ldots, b_n)$ is a quasi-polynomial in $b_1^2, b_2^2, \ldots, b_n^2$ of degree $3g - 3 + n$.

EXAMPLES OF LATTICE POINT POLYNOMIALS

If $\mathbf{b} = b_1 + b_2 + \cdots + b_n$ is odd, then $N_{g,n}(b_1, b_2, \ldots, b_n) = 0$. To simplify life, let's only consider the case when all of $b_1, b_2, \ldots, b_n$ are even.

g	n	$N_{g,n}(b_1, b_2, \ldots, b_n)$
0	3	1
1	1	$\frac{1}{48}(b_1^2 - 4)$
0	4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4)$
1	2	$\frac{1}{384}(b_1^2 + b_2^2 - 4)(b_1^2 + b_2^2 - 8)$
2	1	$\frac{1}{2^{16} \times 3^3 \times 5}(b_1^2 - 4)(b_1^2 - 16)(b_1^2 - 36)(5b_1^2 - 32)$
3	1	$\frac{1}{2^{25} \times 3^6 \times 5^2 \times 7}(5b_1^4 - 188b_1^2 + 1152)\prod_{k=1}^{5}(b_1^2 - 4k^2)$

Question

What do the coefficients mean?

COEFFICIENTS OF LATTICE POINT POLYNOMIALS

Theorem [Norbury, 2010]

- If $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 3g - 3 + n$, then the coefficient of the top degree term $\frac{b_1^{2\alpha_1} b_2^{2\alpha_2} \cdots b_n^{2\alpha_n}}{\alpha_1! \alpha_2! \cdots \alpha_n!}$ in $N_{g,n}(b_1, b_2, \ldots, b_n)$ is $\frac{1}{2^{5g-6+2n}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}$.
- The constant term is given by $N_{g,n}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,n})$.

Proof.

- The lattice point polynomial $N_{g,n}$ approximates the volume of the moduli space, up to a constant factor. Kontsevich and Mirzakhani tell us that this volume stores psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- Consider the following meromorphic function and calculate its value at infinity in two ways.

$$\sum_{b_1, b_2, \ldots, b_n = 1}^{\infty} N_{g,n}(b_1, b_2, \ldots, b_n)\, z^{b_1 + b_2 + \cdots + b_n}$$

□

Question

What do the intermediate coefficients mean?

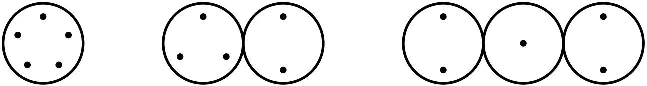
LATTICE POINTS IN COMPACTIFIED MODULI SPACES

New idea

Count lattice points in compactified moduli spaces of curves

Example

Points in $\overline{\mathcal{M}}_{0,5}$ represent curves of the following types.



$$\overline{\mathcal{M}}_{0,5} = \underset{\text{1 labelling}}{\mathcal{M}_{0,5}} \cup \underset{\text{10 labellings}}{\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}} \cup \underset{\text{15 labellings}}{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}$$

$$\begin{aligned}\overline{N}_{0,5}(b_1, b_2, b_3, b_4, b_5) &= N_{0,5}(b_1, b_2, b_3, b_4, b_5) \\ &+ \sum_{\text{10 terms}} N_{0,4}(b_i, b_j, b_k, 0) \cdot N_{0,3}(b_\ell, b_m, 0) \\ &+ \sum_{\text{15 terms}} N_{0,3}(b_i, b_j, 0) \cdot N_{0,3}(b_k, 0, 0) \cdot N_{0,3}(b_\ell, b_m, 0)\end{aligned}$$

COMPACTIFIED LATTICE POINT POLYNOMIALS

Fact

- The enumeration $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is a quasi-polynomial in $b_1^2, b_2^2, \ldots, b_n^2$ of degree $3g - 3 + n$.
- The quasi-polynomials $N_{g,n}$ and $\overline{N}_{g,n}$ agree to leading order — so the top degree part of $\overline{N}_{g,n}$ stores all psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- The constant term is given by $\overline{N}_{g,n}(0, 0, \ldots, 0) = \chi(\overline{\mathcal{M}}_{g,n})$.

Theorem

There exists a recursion which can be used to effectively compute $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$. Again, $\overline{N}_{g,n}$ relies on $\overline{N}_{g-1,n+1}$, $\overline{N}_{g,n-1}$, and $\overline{N}_{g_1,n_1} \times \overline{N}_{g_2,n_2}$ for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$.

Corollary

If $\chi_{g,n}$ denotes the orbifold Euler characteristic of $\overline{\mathcal{M}}_{g,n}$, then

$$\chi_{g,n+1} = (2 - 2g - n)\, \chi_{g,n} + \frac{1}{2}\, \chi_{g-1,n+2} + \frac{1}{2} \sum_{h=0}^{g} \sum_{k=0}^{n} \binom{n}{k}\, \chi_{h,k+1}\, \chi_{g-h,n-k+1}.$$

EXAMPLES OF COMPACTIFIED LATTICE POINT POLYNOMIALS

g	n	$\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$
0	3	1
1	1	$\frac{1}{48}(b_1^2 + 20)$
0	4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
1	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2b_2^2 + 48b_1^2 + 48b_2^2 + 192)$
0	5	$\frac{1}{32}\sum b_i^4 + \frac{1}{8}\sum b_i^2 b_j^2 + \frac{7}{8}\sum b_i^2 + 7$
2	1	$\frac{1}{1769472}b_1^8 + \frac{3}{40960}b_1^6 + \frac{133}{61440}b_1^4 + \frac{1087}{34560}b_1^2 + \frac{247}{1440}$

Question

What do the intermediate coefficients mean and are they always positive?

FROM RIBBON GRAPHS TO STABLE MAPS

Fact

Let $Z_{g,n}(b_1, b_2, \ldots, b_n) \subseteq \mathcal{M}_{g,n}(\mathbb{P}^1, \mathbf{b})$ denote the set of maps which satisfy

- $f^{-1}(\infty) = \{p_1, p_2, \ldots, p_n\}$ with ramification of order b_k at p_k
- each point in $f^{-1}(1)$ has ramification of order 2; and
- there is no point with ramification of order 1 over $0 \in \mathbb{P}^1$.

Then $N_{g,n}(b_1, b_2, \ldots, b_n) = \# Z_{g,n}(b_1, b_2, \ldots, b_n)$.

Fact

Let $\overline{Z}_{g,n}(b_1, b_2, \ldots, b_n) \subseteq \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \mathbf{b})$ denote the set of stable maps which satisfy

- $f^{-1}(\infty) = \{p_1, p_2, \ldots, p_n\}$ with ramification of order b_k at p_k
- each point in $f^{-1}(1)$ has ramification of order 2; and
- every point with ramification of order 1 over $0 \in \mathbb{P}^1$ is a node.

Then $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \chi\left(\overline{Z}_{g,n}(b_1, b_2, \ldots, b_n)\right)$.

Question

Is $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ a Gromov–Witten invariant?

RIBBON GRAPHS AS HURWITZ PROBLEMS

Think of a ribbon graph as a bunch of half-edges. There is a permutation σ_V of the half-edges which rotates about a vertex, a permutation σ_E which swaps coupled half-edges, and a permutation σ_F which rotates about a face.

So $N_{g,n}(b_1, b_2, \ldots, b_n)$ counts triples $(\sigma_V, \sigma_E, \sigma_F)$ such that

- $\sigma_V \sigma_E \sigma_F = \text{id}$;
- σ_F has cycle type $(b_1, b_2, \ldots, b_n)$;
- σ_E has cycle type $(2, 2, \ldots, 2)$;
- σ_V has $\frac{1}{2}\mathbf{b} + 2 - 2g - n$ cycles;
- σ_V has no fixed points; and
- $\langle \sigma_V, \sigma_E, \sigma_F \rangle = S_{\mathbf{b}}$.

GW/H correspondence [Okounkov–Pandharipande, 2006]

Hurwitz problems can be compactified to give Gromov–Witten invariants of curves. This involves replacing the permutations in the Hurwitz problem with completed permutations, which arise in the representation theory of symmetric groups.

Question

Does $\overline{N}_{g,n}$ arise by treating $N_{g,n}$ as a Hurwitz problem and using completed permutations?

EYNARD–ORANTIN TOPOLOGICAL RECURSION

- INPUT: A Riemann surface C with two meromorphic functions x and y.
- OUTPUT: Meromorphic multilinear forms $\omega_{g,n}(z_1, z_2, \ldots, z_n)$ on C.
- RULE: These multilinear forms satisfy a recursion where $\omega_{g,n}$ is a certain residue over $\omega_{g-1,n+1}, \omega_{g,n-1}$ and $\omega_{g_1,n_1} \times \omega_{g_2,n_2}$ for $g_1 + g_2 = g$ and $n_1 + n_2 = n + 1$.

This recursion works for the psi-class intersection numbers, Mirzakhani's Weil–Petersson volumes, simple Hurwitz numbers, plane partitions, matrix models, and conjecturally for Gromov–Witten invariants of any Calabi–Yau threefold.

Theorem

The multilinear forms associated to the curve $xy - y^2 = 1$ are given by

$$\begin{aligned} \omega_{g,n}(z_1, z_2, \ldots, z_n) &= \sum_{b_1, b_2, \ldots, b_n = 1}^{\infty} N_{g,n}(b_1, b_2, \ldots, b_n) \prod_{k=1}^{n} b_k z_k^{b_k - 1} \, dz_k \\ \omega_{g,0} &= \chi(\mathcal{M}_{g,0}) \end{aligned}$$

Question

Is there a topological recursion for $\overline{N}_{g,n}$?

RANDOM QUESTIONS AND ANSWERS

- Isn't it strange to exclude degree 1 vertices?
  Yes — including degree 1 vertices gives polynomials in $b_1, b_2, \ldots, b_n$ and introduces combinatorial factors. It appears that the coefficients are still positive and possibly nicer than those of $\overline{N}_{g,n}$.

- Kontsevich defined stable ribbon graphs — are they what we're counting?
  Yes — but the weight of a stable ribbon graph must include a factor of $\chi(\mathcal{M}_{g,n})$ for collapsed components.

- Why are there so many proofs of the Witten–Kontsevich theorem?
  Here's one explanation — the Eynard–Orantin topological recursion for $y = x^2$ produces psi-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$. In some sense, this is a universal local picture.

THANKS

If you would like more information, you can

- find the slides at http://www.ms.unimelb.edu.au/~nndo
- download the paper at http://arxiv.org/abs/1012.5923
- email me at normdo@gmail.com
- speak to me at the front of the lecture theatre.