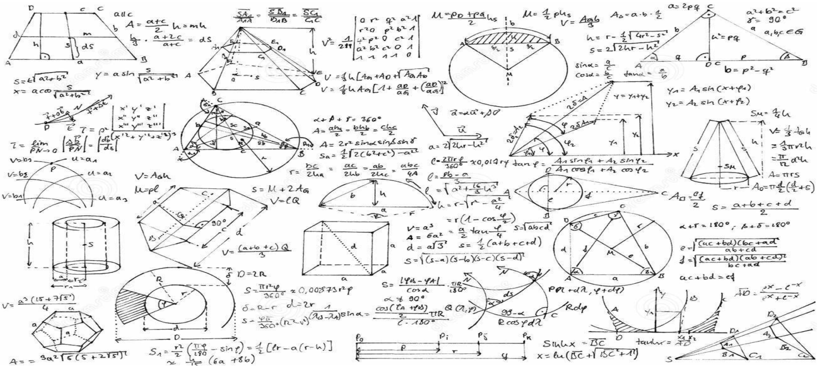
THE HITCHHIKER'S GUIDE TO GEOMETRY

AMSI Public Lecture @ The University of Newcastle

16 January 2015

Norm Do

Monash University



IN THE BEGINNING. . .

Euclid said, “Let there be axioms!"

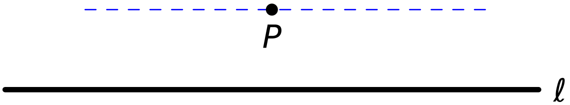
Ruler. You can draw a line segment between two points.

Long ruler. You can extend a line segment indefinitely.

Compass. You can draw a circle with a line segment as radius.

Set square. All right angles are the same.

Parallel postulate. Given a point P not on a line ℓ, you can draw **exactly one line** through P parallel to ℓ.



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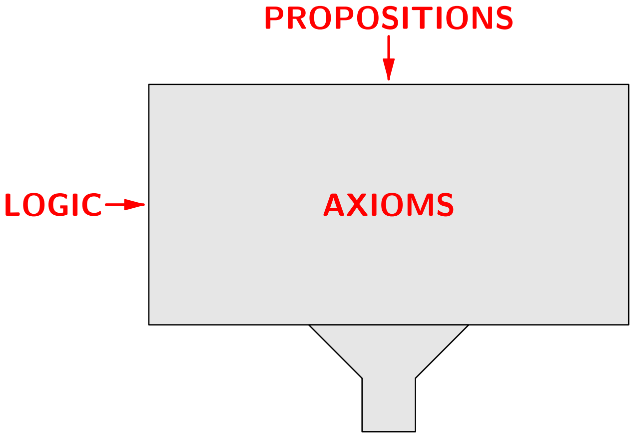
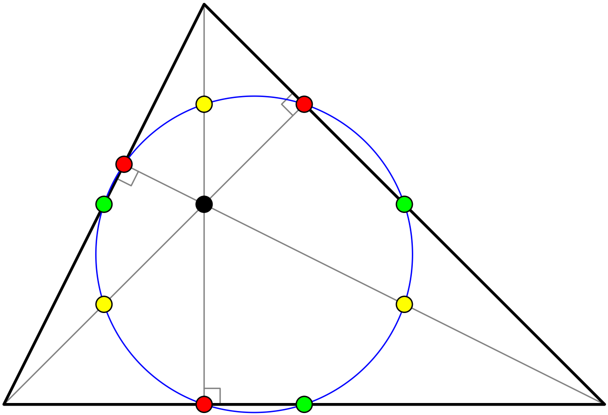


Euclid’s Elements, Book 1, Proposition 32

The sum of the angles in a triangle is 180°.

A geometric gem



This is the nine-point circle!

Was Euclid wrong?

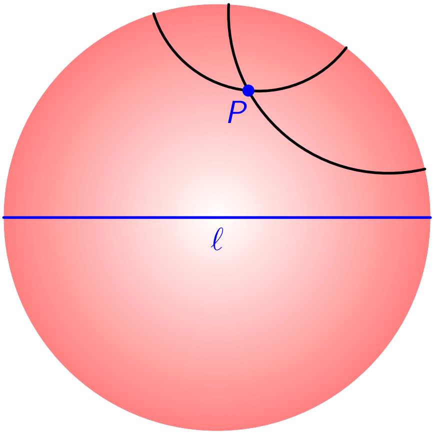
Could there be geometry that doesn't satisfy Euclid's axioms?

Yes, just consider the geometry of the Earth!

You can draw **no lines** through Paris parallel to the equator.

How wrong was Euclid?

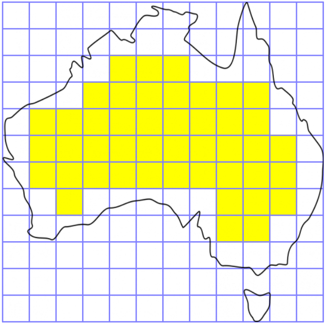
This circular pond of quicksand is called the hyperbolic plane.



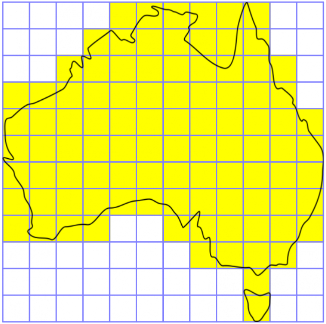
You can draw **many lines** through P parallel to ℓ.

MEASUREMENT

What is the area of Australia?

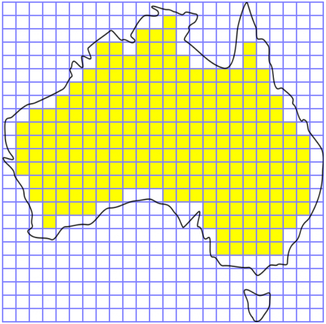


underestimate

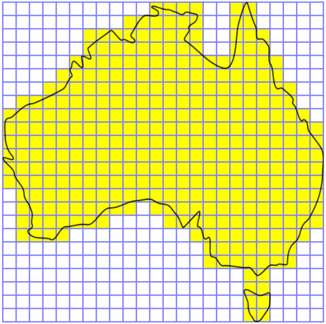


overestimate

What is the area of Australia?



better underestimate



better overestimate

Can we calculate area finitely?

Two polygons are called scissors congruent if you can cut one into polygons that can be rearranged to give the other.

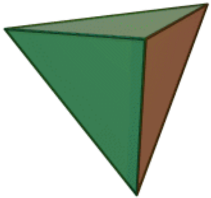
Wallace–Bolyai–Gerwien theorem

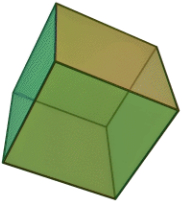
Two polygons are scissors congruent if and only if they have the same area.

Can we calculate **volume** finitely?

Hilbert’s third problem

Are two polyhedra with the same volume scissors congruent?

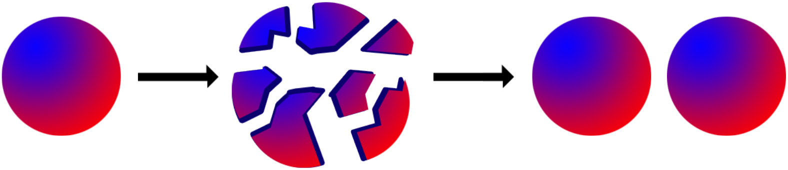




Dehn’s theorem

A regular tetrahedron and a cube are **not** scissors congruent.

The Banach–Tarski paradox

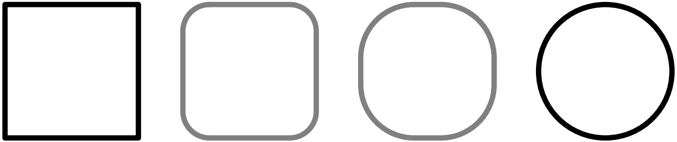


You can divide the points of a unit sphere into five “groups”, then move those groups around to recreate two unit spheres!

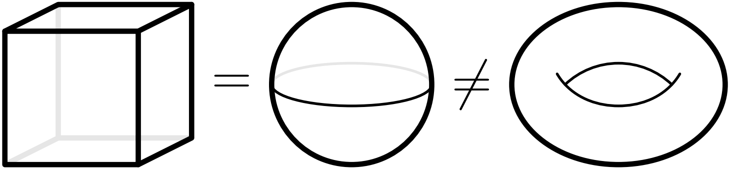

BANACH-TARSKI
DUPLA-SHRINKER

TOPOLOGY

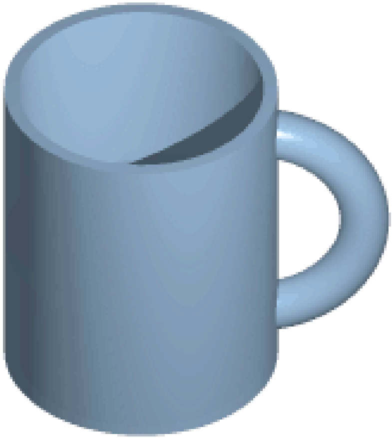
Coffee cups = doughnuts?



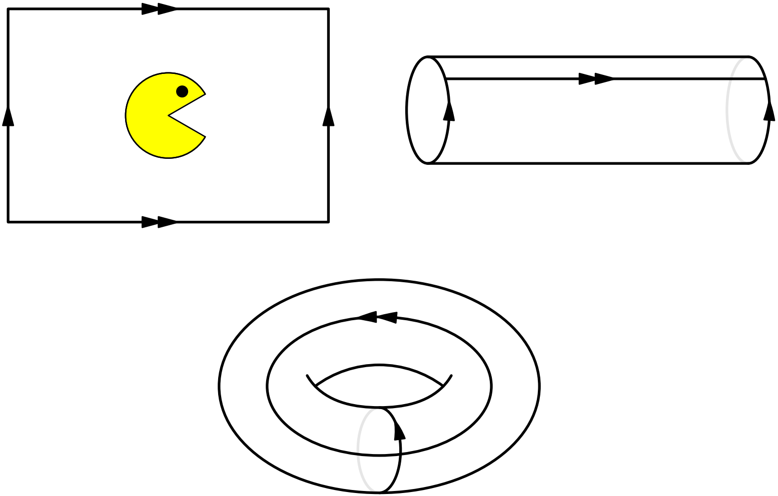
These are all the same in topology!



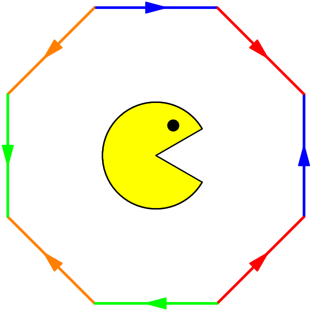
Coffee cups = doughnuts!



Pac-Man world

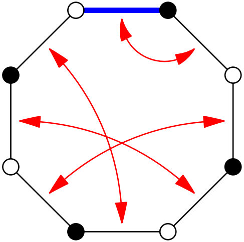


Other Pac-Man worlds



My latest problem

Question. How many ways are there to glue a black-and-white polygon with $2n$ sides and obtain a surface with h holes?



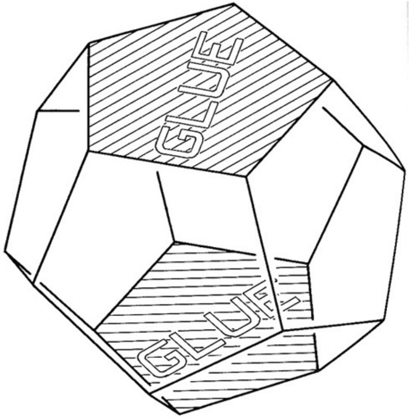
Answer. If we call the answer $F(h, n)$, then

$$F(h,n) = \frac{2(2n-1)}{n+1} F(h, n-1) + \frac{(n-1)^2(n-2)}{n+1} F(h-1, n-2).$$

What if Pac-Man lived in three dimensions?

Mathematical 3-dimensional universes are called 3-manifolds.

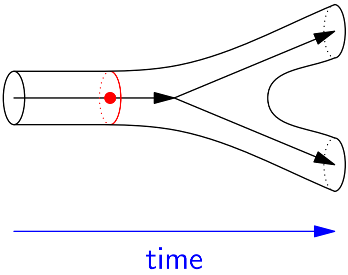
Here is Poincaré dodecahedral space, which we might live in!




DODECAHEDRON

GEOMETRY MEETS "PHYSICS"

Particles vs. strings



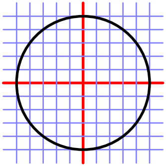
Counting curves

Euclid's 1st axiom. Exactly **1 line** passes through **2 points**.

Question. How many rational curves of degree d pass through $3d-1$ points?

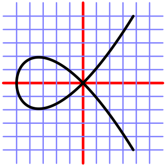
Here is a degree 2 curve.

$$x = \frac{1-t^2}{1+t^2} \qquad y = \frac{2t}{1+t^2}$$



Here is a degree 3 curve.

$$x = \frac{t^2-1}{1} \qquad y = \frac{t^3-t}{1}$$

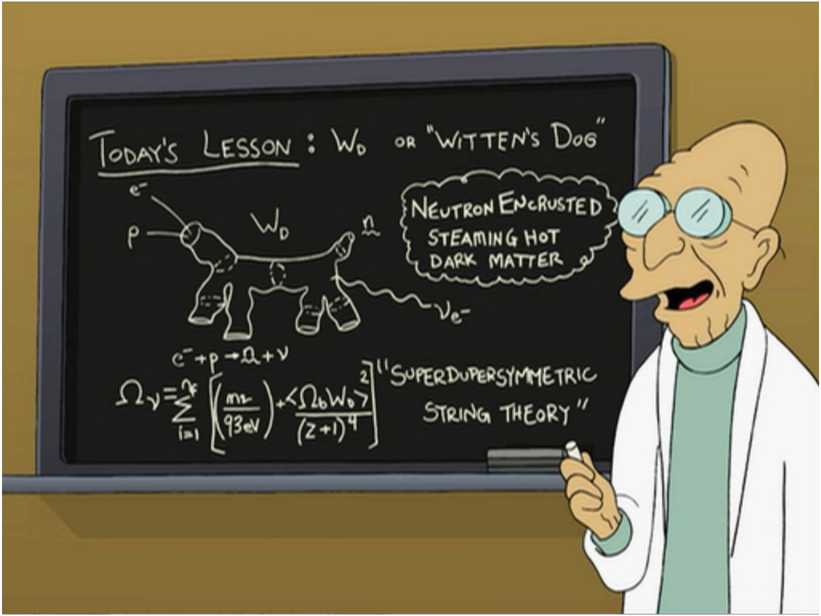


String theory provides the answer!

d	N_d	discoverer
1	1	—
2	1	—
3	12	Steiner, 1848
4	620	Zeuthen, 1873
5	87,304	Kontsevich, 1995
6	26,312,976	Kontsevich, 1995
$\vdots$	$\vdots$	$\vdots$

Kontsevich's formula

$$N_d = \sum_{a+b=d} N_a N_b \left[a^2 b^2 \binom{3d-4}{3a-2} - a^3 b \binom{3d-4}{3a-1} \right]$$


TODAY'S LESSON : W_D OR "WITTEN'S DOG"
e^-
p
W_D
NEUTRON ENCRUSTED STEAMING HOT DARK MATTER
ν_{e^-}
$e^- + p \to \Omega + \nu$
"SUPERDUPERSYMMETRIC STRING THEORY"