A new path to Witten's conjecture via hyperbolic geometry

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In proving Witten's conjecture, Kontsevich produced an amazing combinatorial formula relating intersection numbers on moduli spaces of curves with enumeration of ribbon graphs. Recently, Mirzakhani gave an entirely different proof by relating these intersection numbers to the volumes of moduli spaces of hyperbolic surfaces. In this talk, I will outline how Kontsevich's combinatorial formula emerges naturally from studying the asymptotics of Mirzakhani's volumes.

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What is Witten's conjecture?

- Moduli space of stable curves with genus g and n marked points

$$\overline{\mathcal{M}}_{g,n}$$

- Psi-classes on $\overline{\mathcal{M}}_{g,n}$

$$\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

- Intersection numbers of psi-classes

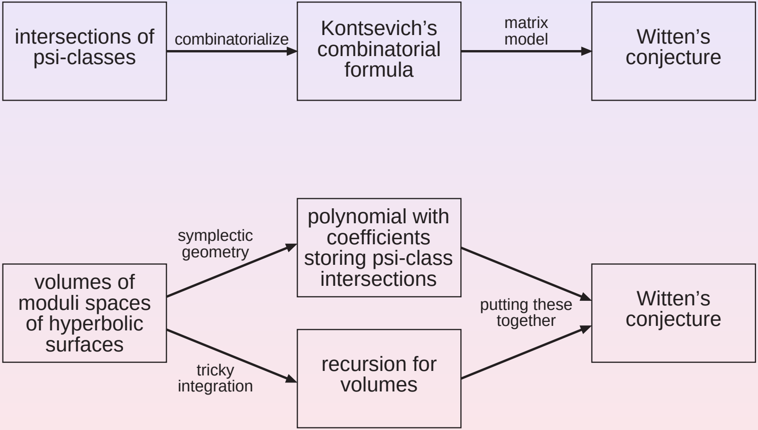
$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \ldots \psi_n^{\alpha_n} \in \mathbb{Q}^+ \text{ where } |\alpha| = 3g - 3 + n$$

Witten's conjecture

I can find a generating function for these intersection numbers and a system of differential equations (the KdV hierarchy) which this generating function satisfies. These uniquely determine all intersection numbers of psi-classes.

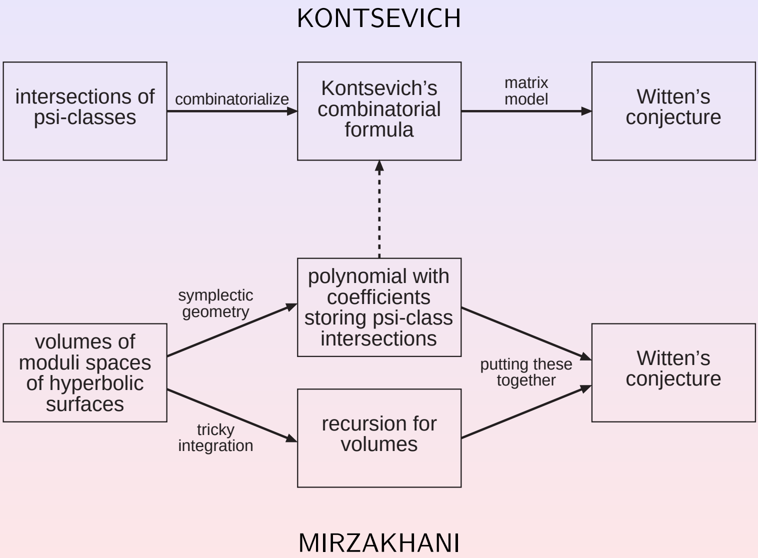
Kontsevich's proof and Mirzakhani's proof

KONTSEVICH



MIRZAKHANI

Kontsevich's proof and Mirzakhani's proof



Kontsevich's combinatorial formula

Kontsevich's combinatorial formula

$$\sum_{|\alpha|=3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \prod_{k=1}^{n} \frac{(2\alpha_k - 1)!!}{s_k^{2\alpha_k+1}} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|\mathrm{Aut}(\Gamma)|} \prod_{e \in \Gamma} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

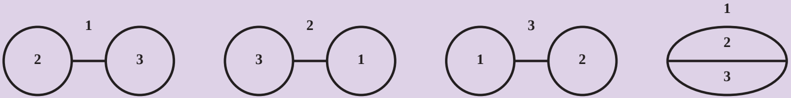
- LHS: polynomial in $\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_n}$
  - coefficients store all intersection numbers on $\overline{\mathcal{M}}_{g,n}$
- RHS: rational polynomial in $s_1, s_2, \ldots, s_n$
  - outside: sum over trivalent ribbon graphs Γ of type (g, n)
  - inside: product over edges of Γ
  - between: a constant
- Remarks
  - $|\mathrm{Aut}(\Gamma)|$ is almost always 1
  - $\ell(e)$ and $r(e)$ are the labels on "the left" and on "the right" of e
  - Kontsevich's combinatorial formula is unbelievable!

What is a ribbon graph?

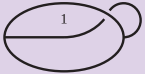
A ribbon graph of type (g, n) is

- a graph with a cyclic ordering of the half-edges at every vertex
- which can be thickened to give an oriented surface of genus g and
- n boundary components labelled from 1 up to n.

Trivalent ribbon graphs of type $(0, 3)$



Trivalent ribbon graph of type $(1, 1)$



Kontsevich's combinatorial formula at work

- Consider $g = 0$ and $n = 3$.
- The LHS is easy!

$$LHS = \frac{\int_{\overline{\mathcal{M}}_{0,3}} \psi_1^0 \psi_2^0 \psi_3^0}{s_1 s_2 s_3}$$

- For the RHS, we obtain one term for each of the four trivalent ribbon graphs of type $(0, 3)$.

$$\begin{aligned} RHS &= \frac{2}{2s_1(s_1+s_2)(s_1+s_3)} + \frac{2}{2s_2(s_2+s_3)(s_2+s_1)} \\ &\quad + \frac{2}{2s_3(s_3+s_1)(s_3+s_2)} + \frac{2}{(s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{s_2 s_3(s_2+s_3) + s_3 s_1(s_3+s_1) + s_1 s_2(s_1+s_2) + 2s_1 s_2 s_3}{s_1 s_2 s_3(s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{(s_1+s_2)(s_2+s_3)(s_3+s_1)}{s_1 s_2 s_3(s_1+s_2)(s_2+s_3)(s_3+s_1)} \\ &= \frac{1}{s_1 s_2 s_3} \end{aligned}$$

- Conclusion: $\int_{\overline{\mathcal{M}}_{0,3}} \psi_1^0 \psi_2^0 \psi_3^0 = 1$

From algebraic curves to hyperbolic surfaces

algebraic curves with genus g and n marked points	$\Longleftrightarrow$	Riemann surfaces with genus g and n marked points	$\Longleftrightarrow$	hyperbolic surfaces with genus g and n cusps
				$\Updownarrow$
				hyperbolic surfaces with genus g and n geodesic boundary components

- Let $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$ denote the moduli space of hyperbolic surfaces with geodesic boundaries of lengths $L_1, L_2, \ldots, L_n$.
- It has the Weil-Petersson symplectic form $\omega \ldots$
- $\ldots$ and the Weil-Petersson volume form $\frac{\omega^{3g-3+n}}{(3g-3+n)!}$.
- Remark: As the geodesic boundary lengths vary, the symplectic structure varies.

Volumes of moduli spaces

Question

What is the volume of $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$ with respect to the Weil-Petersson volume form?

Answer (Mirzakhani)

The volume $V_{g,n}(L_1, L_2, \ldots, L_n)$ of $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$ is given by the following formula.

$$\sum_{|\alpha|+m=3g-3+n} \frac{(2\pi)^m}{2^{|\alpha|}\alpha!m!} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \ldots \psi_n^{\alpha_n} \kappa_1^m \right) L_1^{2\alpha_1} L_2^{2\alpha_2} \ldots L_n^{2\alpha_n}$$

- This is a polynomial in $L_1, L_2, \ldots, L_n$.
- Its coefficients store all intersection numbers of psi-classes...
- ...and the κ_1 class on $\overline{\mathcal{M}}_{g,n}$.

Why volume asymptotics?

Kontsevich's combinatorial formula (again)

$$\sum_{|\alpha|=3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \prod_{k=1}^{n} \frac{(2\alpha_k - 1)!!}{s_k^{2\alpha_k+1}} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|\mathrm{Aut}(\Gamma)|} \prod_{e \in \Gamma} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

INTERSECTION NUMBERS $\leftrightarrow$ RIBBON GRAPHS

Volume asymptotics and the LHS

The intersection numbers of psi-classes are stored in the top degree terms of $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$. The top degree terms arise when we consider the volume polynomial as the lengths $L_1, L_2, \ldots, L_n$ approach infinity.

Volume asymptotics and the RHS

As the lengths $L_1, L_2, \ldots, L_n$ approach infinity, the elements of $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$ look more and more like ribbon graphs.

Volume asymptotics

- Take the asymptotic limit...

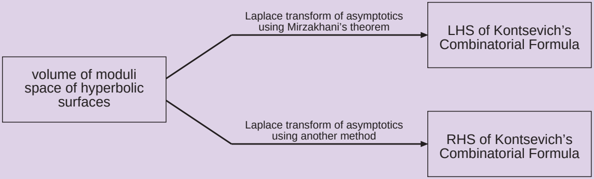
$$\lim_{N\to\infty} \frac{V_{g,n}(Nx_1, Nx_2, \dots, Nx_n)}{N^{6g-6+2n}} = \sum_{|\alpha|=3g-3+n} \frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n}}{2^{3g-3+n}\alpha!} x_1^{2\alpha_1} x_2^{2\alpha_2} \dots x_n^{2\alpha_n}$$

- ...then take the Laplace transform

$$\begin{aligned}
&\mathcal{L}\left\{ \sum_{|\alpha|=3g-3+n} \frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n}}{2^{3g-3+n}\alpha!} x_1^{2\alpha_1} x_2^{2\alpha_2} \dots x_n^{2\alpha_n} \right\} \\
= & \sum_{|\alpha|=3g-3+n} \frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n}}{2^{3g-3+n}\alpha!} \prod_{k=1}^{n} \mathcal{L}\left\{ x_k^{2\alpha_k} \right\} \\
= & \sum_{|\alpha|=3g-3+n} \frac{\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n}}{2^{3g-3+n}\alpha!} \prod_{k=1}^{n} \frac{(2\alpha_k)!}{s_k^{2\alpha_k+1}} \\
= & \sum_{|\alpha|=3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \prod_{k=1}^{n} \frac{(2\alpha_k-1)!!}{s_k^{2\alpha_k+1}} \\
= & \text{ LHS of Kontsevich's Combinatorial Formula}
\end{aligned}$$

From volumes to Kontsevich's combinatorial formula

Our approach to Kontsevich's combinatorial formula



$$\text{RHS of Kontsevich's combinatorial formula} = \sum_{\Gamma \in TRG_{g,n}} \frac{2^{2g-2+n}}{|\mathrm{Aut}(\Gamma)|} \prod_{e \in \Gamma} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

Questions

- Why do we get a sum over trivalent ribbon graphs?
- Why do we get a product over the edges of the graph?
- How can we calculate the constant in front of the product?

Divide and conquer

- We can assign a ribbon graph of type (g, n) to a hyperbolic surface of genus g with n geodesic boundary components.

$$\mathcal{M}_{g,n}(Nx_1, Nx_2, \ldots, Nx_n) = \bigcup_{\Gamma \in RG_{g,n}} \mathcal{M}^{\Gamma}_{g,n}(Nx_1, Nx_2, \ldots, Nx_n)$$

$$V_{g,n}(Nx_1, Nx_2, \ldots, Nx_n) = \sum_{\Gamma \in TRG_{g,n}} V^{\Gamma}_{g,n}(Nx_1, Nx_2, \ldots, Nx_n)$$

- We only see trivalent ribbon graphs because they are generic.

This explains why we get a sum over trivalent ribbon graphs.

- It now suffices to prove the following for a trivalent ribbon graph Γ of type (g, n).

$$\mathcal{L}\left\{\lim_{N\to\infty} \frac{V^{\Gamma}_{g,n}(Nx_1, Nx_2, \ldots, Nx_n)}{N^{6g-6+2n}}\right\} = 2^{2g-2+n} \prod_{e\in\Gamma} \frac{1}{s_{\ell(e)} + s_{r(e)}}$$

From hyperbolic surfaces to ribbon graphs

Key geometric observation

In the $N \to \infty$ limit, the elements of $\mathcal{M}_{g,n}(Nx_1, Nx_2, \ldots, Nx_n)$ become *metric* ribbon graphs of type (g, n). Metric means that there is a positive length associated to each edge.

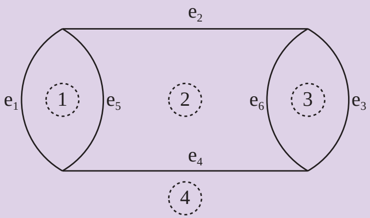
- To calculate the volume $V^{\Gamma}_{g,n}(Nx_1, Nx_2, \ldots, Nx_n)$ in the $N \to \infty$ limit, we need
  - local coordinates,
  - the region of integration, and
  - the volume form.
- Once we fix a trivalent ribbon graph Γ of type (g, n), we have local coordinates $(e_1, e_2, \ldots, e_{6g-6+3n})$, corresponding to the edge lengths of the metric ribbon graphs.

Region of integration

- In the $N \to \infty$ limit, the region of integration consists of all tuples $(e_1, e_2, \ldots, e_{6g-6+3n}) \in \mathbb{R}_+^{6g-6+3n}$ which satisfy n linear constraints, one for each boundary component.
- Therefore, in the coordinates, $e_1, e_2, \ldots, e_{6g-6+3n}$, the region of integration is a polytope.
- Check: the dimension of this region is

$$(6g - 6 + 3n) - n = 6g - 6 + 2n.$$

Example



$$\begin{aligned} Nx_1 &= e_1 + e_5 \\ Nx_2 &= e_2 + e_4 + e_5 + e_6 \\ Nx_3 &= e_3 + e_6 \\ Nx_4 &= e_1 + e_2 + e_3 + e_4 \end{aligned}$$

Volume form

Theorem (Wolpert)

Suppose that the lengths $\ell_1, \ell_2, \ldots, \ell_{6g-6+2n}$ of simple closed geodesics $\gamma_1, \gamma_2, \ldots, \gamma_{6g-6+2n}$ give local coordinates for $\mathcal{M}_{g,n}(L_1, L_2, \ldots, L_n)$. Then the Weil-Petersson volume form is

$$\frac{1}{\sqrt{\det W}} \; d\ell_1 \wedge d\ell_2 \wedge \ldots \wedge d\ell_{6g-6+2n}$$

where $W_{ij} = \sum_{p \in \gamma_i \cap \gamma_j} \cos\theta_p$.

- In the $N \to \infty$ limit, W is a constant integer matrix since

$$\theta_p = 0 \text{ or } \pi \Rightarrow \cos\theta_p = 1 \text{ or } -1.$$

- We can choose γ so that $d\ell = 2de$.

Fact

In the $N \to \infty$ limit, the Weil-Petersson volume form can be written as

$$C \;\; de_1 \wedge de_2 \wedge \ldots \wedge de_{6g-6+2n}.$$

Volume calculation

- Good news: Using the coordinates $e_1, e_2, \ldots, e_{6g-6+3n}$, the region of integration AND the volume form are simple!
- Now we want to compute the volume of a polytope.

Volumes of polytopes

Suppose that A is an $M \times N$ matrix with non-negative entries and rank M. Let $S(\mathbf{x}) = \{\mathbf{e} \in \mathbb{R}_+^N \mid A\mathbf{e} = \mathbf{x}\}$ and let $V(\mathbf{x})$ be the volume of $S(\mathbf{x})$ with respect to $de_{M+1} \wedge de_{M+2} \wedge \ldots \wedge de_N$. Then

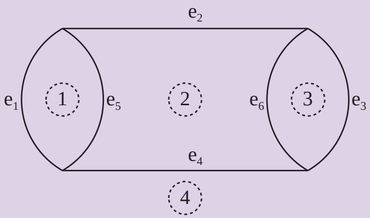
$$\mathcal{L}\{V(\mathbf{x})\} = \frac{D}{\prod_{k=1}^{N} [A^t \mathbf{s}]_k}$$

where $D = \det[A_{ij}]_1^M$.

This explains why we get a product over the edges of the graph.

Volume calculation example

Example



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$$\begin{aligned} & \mathcal{L}\left\{ \lim_{N\to\infty} V^{\Gamma}_{0,4}(Nx_1, Nx_2, Nx_3, Nx_4)/N^2 \right\} \\ = \; & \mathcal{L}\{C \times V(Nx_1, Nx_2, Nx_3, Nx_4)/N^2\} \\ = \; & C \times \mathcal{L}\{V(x_1, x_2, x_3, x_4)\} \\ = \; & \frac{C \times D}{(s_1+s_4)(s_2+s_4)(s_3+s_4)(s_2+s_4)(s_1+s_2)(s_2+s_3)} \end{aligned}$$

Combinatorics of the constant

- How can we calculate the constant in front of the product?
  - Take the constant in the volume form,
  - take the constant in the volume calculation, and
  - put these together to get...

NORM's COMBINATORICS PROBLEM

Consider a ribbon graph of type (g, n). Colour n of the edges blue and the remaining $6g - 6 + 2n$ edges red. Let A be the $n \times n$ matrix formed from the adjacency between the blue edges and the faces. Let B be the $(6g - 6 + 2n) \times (6g - 6 + 2n)$ skew-symmetric matrix formed from the oriented adjacency between the red edges. Then

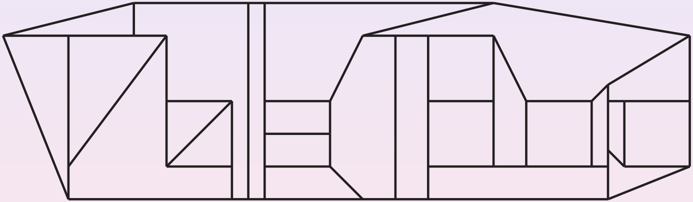
$$\det B = 2^{2g-2}(\det A)^2.$$

This will explain how we can calculate the constant in front of the product.

Example

The following is a ribbon graph with

$$g = 0 \quad \text{and} \quad n = 26.$$



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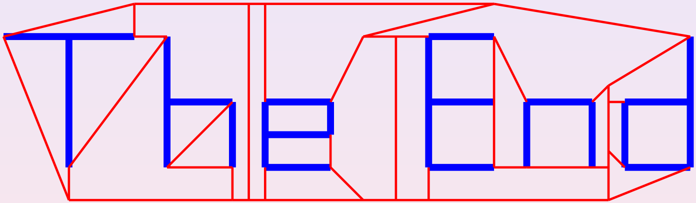

$$\begin{array}{cccccc} \det B & = & 2^{2g-2} & \times & (\det A)^2 \\ 256 & = & \frac{1}{4} & \times & 32^2 \end{array}$$