THE GEOMETRY AND COMBINATORICS OF MODULI SPACES

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Norman Do

* Moduli spaces and enumerative geometry

Moduli spaces parametrise geometric objects

different points $\longleftrightarrow$ different objects

nearby points $\longleftrightarrow$ similar objects

Toy example: Moduli space of triangles

$$\mathcal{M}_\Delta = \left\{ (a, b, c) \in \mathbb{R}^3_+ \;\middle|\; \begin{matrix} b+c>a \\ c+a>b \\ a+b>c \end{matrix} \right\} \Big/ S_3$$

Toy question: How many triangles are

$\underbrace{\text{isosceles,}}_{X_{iso} \subseteq \mathcal{M}_\Delta}$ $\underbrace{\text{have a length 5 side,}}_{X_5 \subseteq \mathcal{M}_\Delta}$ and $\underbrace{\text{have a length 7 side}}_{X_7 \subseteq \mathcal{M}_\Delta}$?

Equivalently, what is $\int_{\mathcal{M}_\Delta} X_{iso} \cdot X_5 \cdot X_7 := |X_{iso} \cap X_5 \cap X_7|$?

Cohomology / intersection theory: A naive picture is . . .

spaces $\longrightarrow$  $\longrightarrow$ rings

elements of H^* $\longleftrightarrow$ submanifolds of X

addition $\longleftrightarrow$ formal addition

multiplication $\longleftrightarrow$ intersection

* Moduli spaces of curves

Topology of surfaces:

g=0 g=1 g=2 . . .

Geometry of surfaces:

surfaces

$$\begin{Bmatrix} \text{algebraic} \\ \text{curves} \end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix} \text{Riemann} \\ \text{surfaces} \end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix} \text{hyperbolic} \\ \text{surfaces} \end{Bmatrix}$$

Moduli spaces of curves:

$\mathcal{M}_{g,n} = \left\{ \begin{array}{c} \text{genus } g \text{ smooth algebraic curves} \\ \text{with } n \text{ labelled points} \end{array} \right\}$ e.g.  $\in \mathcal{M}_{2,3}$

$\downarrow$ compactify

$\overline{\mathcal{M}}_{g,n} = \left\{ \begin{array}{c} \text{genus } g \text{ stable algebraic curves} \\ \text{with } n \text{ labelled points} \end{array} \right\}$ e.g.  $\in \overline{\mathcal{M}}_{2,3}$

stable = allow nodes + finitely many automorphisms

Facts:

- $\overline{\mathcal{M}}_{g,n}$ is an orbifold, so intersection numbers are rational
- $\dim \overline{\mathcal{M}}_{g,n} = 2(3g-3+n)$
- $\overline{\mathcal{M}}_{g,n}$ is VERY complicated

Witten-Kontsevich theorem [1991-1992]: We have $\psi_1, \psi_2, \ldots, \psi_n \in H^*(\overline{\mathcal{M}}_{g,n})$ representing natural codimension two submanifolds, where

$$\psi_k = c_1(\underbrace{\mathcal{L}_k}_{\text{cotangent line bundle at } k^{\text{th}} \text{ marked point}})$$

If $a_1 + a_2 + \cdots + a_n = 3g-3+n$, then $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \in \mathbb{Q}$.

Witten says these numbers can be stored in a generating function which satisfies KdV.

Proofs: Kontsevich, Okounkov - Pandharipande, Mirzakhani, ...

* Tiling surfaces

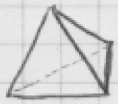
$N_{g,n}(b_1, b_2, \ldots, b_n)$ = # ways to glue edges of labelled polygons with $b_1, b_2, \ldots, b_n$ sides to obtain a genus g surface

= # "lattice points in $\mathcal{M}_{g,n}$"

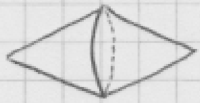
Strebel's theorem:

surface tilings with lengths assigned to edges $\longrightarrow$ points in $\mathcal{M}_{g,n}$

Example: $N_{0,4}(3,3,3,3) = 8$



2 labellings



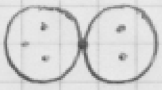
6 labellings

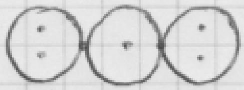
Define $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \#$ "lattice points in $\overline{\mathcal{M}}_{g,n}$"

$= N_{g,n}(b_1, b_2, \ldots, b_n) +$ lower order terms

Example:

$$\overline{\mathcal{M}}_{0,5} = \underbrace{\mathcal{M}_{0,5}}_{\text{1 labelling}} \cup \underbrace{\mathcal{M}_{0,4} \times \mathcal{M}_{0,3}}_{\text{10 labellings}} \cup \underbrace{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}_{\text{15 labellings}}$$





$$\overline{N}_{0,5}(b_1,b_2,b_3,b_4,b_5) = N_{0,5}(b_1,b_2,b_3,b_4,b_5) + \sum_{\text{10 labellings}} N_{0,4}(b_i, b_j, b_k, 0) \times N_{0,3}(b_l, b_m, 0)$$

$$+ \sum_{\text{15 labellings}} N_{0,3}(b_i, b_j, 0) \times N_{0,3}(b_k, 0, 0) \times N_{0,3}(b_l, b_m, 0)$$

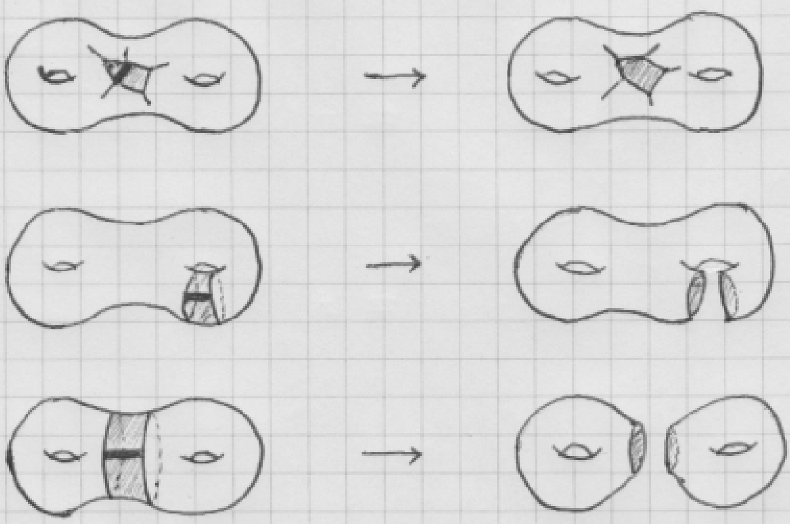
Topological recursion [Do - Norbury, 2011]:

$\overline{N}_{g,n}$ depends on $\overline{N}_{g,n-1}$

$\overline{N}_{g-1,n+1}$

$\overline{N}_{g_1,n_1} \times \overline{N}_{g_2,n_2}$ for $\begin{matrix} g_1 + g_2 = g \\ n_1 + n_2 = n+1 \end{matrix}$

Idea of proof: Remove edges



Theorem [Do-Norbury, 2011]:

- $\overline{N}_{g,n}(b_1, b_2, \ldots, b_n)$ is an even, symmetric quasi-polynomial of degree $2(3g-3+n)$, depending on parity
- (Top degree) If $a_1 + a_2 + \cdots + a_n = 3g-3+n$, then
$$\left[\frac{b_1^{2a_1}}{a_1!} \cdot \frac{b_2^{2a_2}}{a_2!} \cdots \frac{b_n^{2a_n}}{a_n!}\right] \overline{N}_{g,n}(b_1, b_2, \ldots, b_n) = \frac{1}{2^{5g-6+2n}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n}.$$
- (Bottom degree) $\overline{N}_{g,n}(0, 0, \ldots, 0) = \chi(\overline{\mathcal{M}}_{g,n})$

Corollary:

- (Old) Witten-Kontsevich theorem
- (New) Recursion for $\chi_{g,n} := \chi(\overline{\mathcal{M}}_{g,n})$
$$\chi_{g,n} = (3-2g-n)\chi_{g,n-1} + \frac{1}{2}\Big[\chi_{g-1,n+1} + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+1}} \binom{n-1}{n_1-1} \chi_{g_1,n_1} \chi_{g_2,n_2}\Big]$$

Question: Is there geometric meaning for the intermediate coefficients?

Conjecture: Yes — that's why they're positive.

* More pieces of the puzzle

- moduli spaces of curves
$$\left.\begin{array}{c} \text{moduli spaces of curves} \\ \downarrow \\ \text{moduli spaces of maps from curves to } \mathbb{P}^1 \\ \downarrow \\ \text{moduli spaces of maps from curves to } X \end{array}\right\} \text{Gromov - Witten theory}$$

- Eynard-Orantin topological recursion
$$C = \underbrace{\begin{bmatrix}\text{relation between} \\ x \text{ and } y\end{bmatrix}}_{\text{spectral curve}} \longrightarrow \text{EO} \longrightarrow \begin{array}{c}\text{multidifferentials} \\ \omega_{g,n} \text{ on } C\end{array}$$

$\omega_{g,n}$ depends on $\omega_{g,n-1}$

$\omega_{g-1,n+1}$

$\omega_{g_1,n_1} \times \omega_{g_2,n_2}$ for $\begin{array}{l} g_1+g_2=g \\ n_1+n_2=n+1 \end{array}$