COUNTING CURVES ON SURFACES

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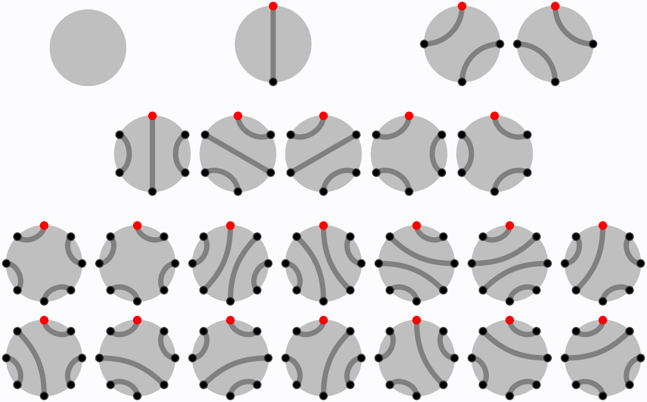
Monash University Discrete Mathematics Seminar
29 February 2016

Choose an even number of points on the boundary of a surface. How many ways are there to pair up these points with disjoint arcs on the surface? The most basic instance of this problem produces the Catalan numbers while the problem in general exhibits a surprisingly rich structure. For example, we will show that this enumeration obeys an effective recursion and exhibits polynomial behaviour. Moreover, there are unexpected connections to algebraic geometry and mathematical physics.

Counting curves creates Catalan

How many ways are there to pair up b points on the boundary of a disk with disjoint arcs?

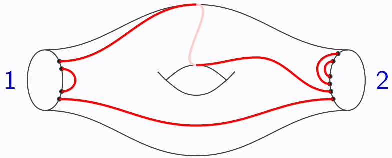


For b even, we get the sequence $1, 1, 2, 5, 14, 42, \ldots$ of Catalan numbers.
For b odd, we get zero.

From disks to surfaces

Surfaces are classified by their genus (g) and number of boundaries (n).

$g = 1$
$n = 2$



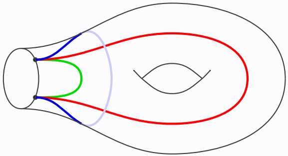
$b_1 = 4$
$b_2 = 6$

Question

Label the boundaries $1, 2, \ldots, n$ and choose b_i points on boundary i.
How many ways are there to pair up these points with disjoint arcs?
Denote the answer by $G_{g,n}(b_1, \ldots, b_n)$.

Example

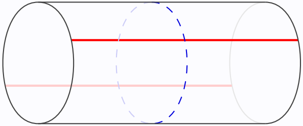
$G_{g,n}(0, \ldots, 0) = 1$
$G_{g,n}(b_1, \ldots, b_n) = 0$ if $\sum b_i$ odd
$G_{0,1}(2m) = \mathrm{Cat}_m = \frac{1}{m+1}\binom{2m}{m}$
$G_{1,1}(2) = 3$

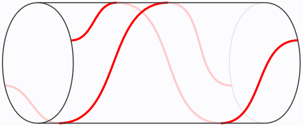


Why isn't the answer infinite?

Two arc diagrams are equivalent if there is a continuous bijection (preserving orientation and boundary points) that takes one to the other.

Equivalently, two arc diagrams are equivalent if they can be related by Dehn twists. (One creates a Dehn twist by cutting along a simple closed curve and gluing back the surface with a 360° twist.)





So equivalent arc diagrams might look different "on the surface"!

Annuli — Insular arc diagrams

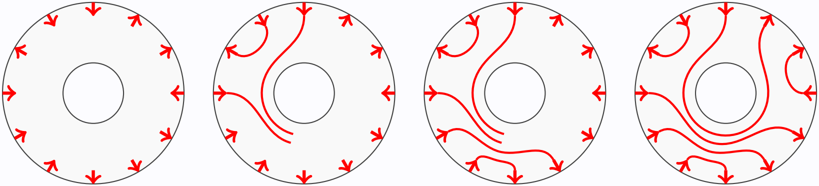
Write $G_{0,2}(b_1, b_2) = \underbrace{T(b_1, b_2)}_{\text{traversing}} + \underbrace{I(b_1, b_2)}_{\text{insular}}$.

Fact

For m a non-negative integer, $I(2m, 0) = \binom{2m}{m}$.

Proof.

There are $\binom{2m}{m}$ ways to draw m arrows inwards and m arrows outwards.



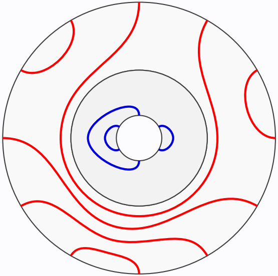
One can draw "anticlockwise" arcs following the arrows uniquely. $\square$

Annuli — Insular arc diagrams (continued)

Corollary

From $I(2m, 0) = \binom{2m}{m}$*, we obtain*

- $I(2m_1, 2m_2) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}$,
- $I(2m_1 + 1, 2m_2 + 1) = 0$,
- $G_{0,1}(2m) = \frac{1}{m+1}\binom{2m}{m}$.



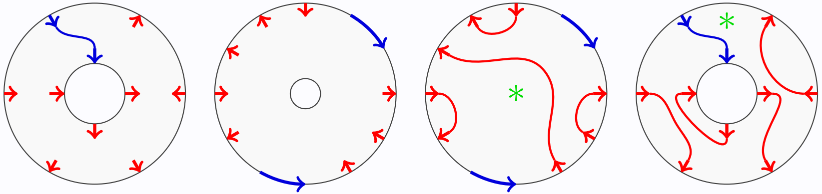
Proof.

- Glue two annuli together to get $I(2m_1, 2m_2) = I(2m_1, 0)\, I(2m_2, 0)$.
- You can't pair up the $2m_1 + 1$ points on boundary 1 with arcs.
- [Przytycki, 1999] For each arc diagram enumerated by $G_{0,1}(2m)$, there are $m + 1$ regions. Punching a hole in one of these regions yields one of the $\binom{2m}{m}$ arc diagrams enumerated by $I(2m, 0)$. $\square$

Annuli — Traversing arc diagrams

Consider the case $(b_1, b_2) = (2m_1, 2m_2)$.

- There are $\binom{2m_1}{m_1}\binom{2m_2}{m_2}$ ways to draw m_i in/out-arrows on boundary i.
- There are $m_1 m_2$ ways to connect an in-arrow on boundary 1 to an out-arrow on boundary 2 by an arc γ.
- Cut along γ to obtain a disk with $m_1 + m_2 - 1$ in/out-arrows.
- Punch a hole in the disk to make an annulus.
- As above, draw "anticlockwise" arcs following the arrows.
- Remove the hole and mark the region it used to be in.
- Glue along γ to obtain an annulus divided into $m_1 + m_2$ regions.



Annuli — Traversing arc diagrams (continued)

We obtain a unique traversing arc diagram with a marked region, so

$$T(2m_1, 2m_2) = \frac{m_1 m_2}{m_1 + m_2} \binom{2m_1}{m_1} \binom{2m_2}{m_2}.$$

A similar argument leads to

$$T(2m_1 + 1, 2m_2 + 1) = \frac{(2m_1 + 1)(2m_2 + 1)}{m_1 + m_2 + 1} \binom{2m_1}{m_1} \binom{2m_2}{m_2}.$$

Theorem

Putting the insular and traversing arc diagrams together yields

$$G_{0,2}(2m_1, 2m_2) = \frac{m_1 m_2 + m_1 + m_2}{m_1 + m_2} \binom{2m_1}{m_1} \binom{2m_2}{m_2}$$

$$G_{0,2}(2m_1 + 1, 2m_2 + 1) = \frac{(2m_1 + 1)(2m_2 + 1)}{m_1 + m_2 + 1} \binom{2m_1}{m_1} \binom{2m_2}{m_2}.$$

Structure theorem

Theorem (Polynomiality)

Let $C(2m) = C(2m+1) = \binom{2m}{m}$. *For* $(g,n) \neq (0,1)$ *or* $(0,2)$,

$$G_{g,n}(b_1, \ldots, b_n) = C(b_1) \cdots C(b_n) \times \widehat{G}_{g,n}(b_1, \ldots, b_n),$$

where $\widehat{G}_{g,n}$ *is a symmetric quasi-polynomial of degree* $3g - 3 + 2n$.

Example

g	n	parity	$\widehat{G}_{g,n}(b_1, \ldots, b_n)$
0	1	(0)	$1/(m_1+1)$
0	2	$(0,0)$	$(m_1m_2+m_1+m_2)/(m_1+m_2)$
0	2	$(1,1)$	$(2m_1+1)(2m_2+1)/(m_1+m_2+1)$
0	3	$(0,0,0)$	$(m_1+1)(m_2+1)(m_3+1)$
0	3	$(1,1,0)$	$(2m_1+1)(2m_2+1)(m_3+1)$
1	1	(0)	$\frac{1}{12}(m^2+5m+12)$

Topological recursion

Theorem

For $S = \{2, 3, \ldots, n\}$ and $b_1 > 0$,

$$G_{g,n}(b_1, \mathbf{b}_S) = \sum_{k \in S} b_k \, G_{g,n-1}(b_1 + b_k - 2, \mathbf{b}_{S \setminus \{k\}})$$

$$+ \sum_{i+j=b_1-2} \left[G_{g-1,n+1}(i, j, \mathbf{b}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} G_{g_1,|I|+1}(i, \mathbf{b}_I) \, G_{g_2,|J|+1}(j, \mathbf{b}_J) \right].$$

Any $G_{g,n}(\mathbf{b})$ can be computed from the initial conditions $G_{g,n}(\mathbf{0}) = 1$.

Remark

The (g, n) case depends on

- $(g, n-1)$,
- $(g-1, n+1)$, and
- $(g_1, n_1) \times (g_2, n_2)$ for $\begin{cases} g_1 + g_2 = g, \\ n_1 + n_2 = n + 1. \end{cases}$

Topological recursion — Sketch proof

What happens when we cut along an arc that meets boundary 1?

- The arc has endpoints on distinct boundaries.

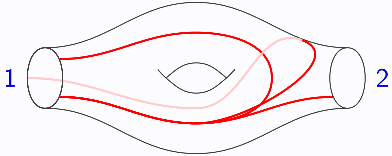
$$(g, n) \rightsquigarrow (g, n-1)$$

- The arc has endpoints on boundary 1 and is "non-separating".

$$(g, n) \rightsquigarrow (g-1, n+1)$$

- The arc has endpoints on boundary 1 and is "separating".

$$(g, n) \rightsquigarrow (g_1, n_1) \times (g_2, n_2)$$



Clean arc diagrams

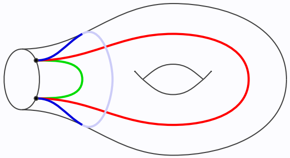
Call an arc diagram clean if there is no arc “parallel to the boundary” (so that cutting along it produces a disk).

Definition

Let $N_{g,n}(b_1, \ldots, b_n)$ be the number of clean arc diagrams on a genus g surface with n labelled boundaries and b_i points chosen on boundary i.

Example

$N_{g,n}(0, \ldots, 0) = 1$
$N_{0,1}(2m) = \delta_{m,0}$
$N_{1,1}(2) = 1$



Clean structure theorem

Theorem

Let $\bar{b} = b + \delta_{b,0}$. *For* $(g, n) \neq (0, 1)$ *or* $(0, 2)$ *and* $\mathbf{b} \neq \mathbf{0}$,

$$N_{g,n}(b_1, \ldots, b_n) = \bar{b}_1 \cdots \bar{b}_n \times \widehat{N}_{g,n}(b_1^2, \ldots, b_n^2),$$

where $\widehat{N}_{g,n}$ *is a symmetric quasi-polynomial of degree* $3g - 3 + n$.

Example

g	n	parity	$\widehat{N}_{g,n}(b_1^2, \ldots, b_n^2)$
0	3	$(0,0,0)$	1
0	3	$(1,1,0)$	1
0	4	$(0,0,0,0)$	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + 2$
0	4	$(1,1,0,0)$	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + \frac{1}{2}$
0	4	$(1,1,1,1)$	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + 2$
1	1	(0)	$\frac{1}{48}(b_1^2 + 20)$

Clean topological recursion

Theorem

For $S = \{2, 3, \ldots, n\}$ and $(g, n) \neq (0, 1)$, $(0, 2)$, $(0, 3)$ and $b_1 > 0$,

$$b_1 \, \widehat{N}_{g,n}(b_1, \mathbf{b}_S) = \sum_{k \in S} \Bigg[\sum_{i+m=b_1+b_k} + \sum_{i+m=b_1-b_k} \Bigg] \frac{\bar{i} \, m}{2} \widehat{N}_{g,n-1}(i, \mathbf{b}_{S \setminus \{k\}})$$

$$+ \sum_{i+j+m=b_1} \frac{\bar{i} \, \bar{j} \, m}{2} \Bigg[\widehat{N}_{g-1,n+1}(i, j, \mathbf{b}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} \widehat{N}_{g_1,|I|+1}(i, \mathbf{b}_I) \, \widehat{N}_{g_2,|J|+1}(j, \mathbf{b}_J) \Bigg]$$

Stable means we exclude terms with $(g, n) = (0, 1)$ or $(0, 2)$.

Proof.

Similar in flavour to the “unclean” topological recursion. $\square$

Prototypical proof of polynomiality

Consider $(g,n)=(0,4)$ and let b_1 be largest.
Recall that $\widehat{N}_{0,3}(b_1,b_2,b_3)=1$ if $b_1+b_2+b_3$ is even.

We have $b_1\widehat{N}_{0,4}(b_1,b_2,b_3,b_4)=\dfrac{1}{2}\displaystyle\sum_{k=2,3,4}A_0(b_1+b_k)+A_0(b_1-b_k)$,

where $A_0(b)=\displaystyle\sum_{\substack{i+m=b\\ m\text{ even}}}\bar{i}\,m=\begin{cases}\frac{1}{12}(b^3+8b) & b\text{ even},\\ \frac{1}{12}(b^3-b) & b\text{ odd}.\end{cases}$

If b_1,b_2,b_3,b_4 are even, then

$$\begin{aligned}\widehat{N}_{0,4}(b_1,b_2,b_3,b_4)&=\frac{1}{24b_1}\Bigg[\sum_{k=2,3,4}(b_1+b_k)^3+8(b_1+b_k)\\&\qquad\qquad+(b_1-b_k)^3+8(b_1-b_k)\Bigg]\\&=\frac{1}{4}(b_1^2+b_2^2+b_3^2+b_4^2)+2.\end{aligned}$$

Prototypical proof of polynomiality (continued)

More generally, we need the following result.

Theorem (Brion–Vergne, 1997)

Let P be a convex lattice polytope in $\mathbb{R}^n$ with non-empty interior I. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a degree d homogeneous polynomial. Then

$$N_P(f,k) = \sum_{\mathbf{x} \in \mathbb{Z}^n \cap kP} f(\mathbf{x}) \qquad \text{and} \qquad N_I(f,k) = \sum_{\mathbf{x} \in \mathbb{Z}^n \cap kI} f(\mathbf{x})$$

are polynomials of degree $n+d$, with $N_I(f,k) = (-1)^{n+d} N_P(f,-k)$.

Corollary

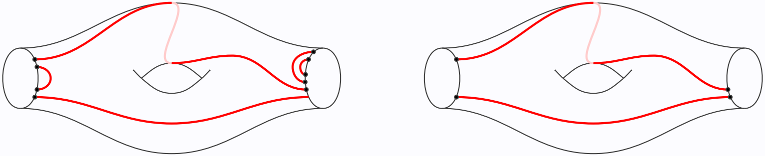
The functions

$$A_m(b) = \sum_{\substack{p+q=b \\ q \text{ even}}} \bar{p}\, p^{2m}\, q \qquad \text{and} \qquad B_{m,n}(b) = \sum_{\substack{p+q+r=b \\ r \text{ even}}} \bar{p}\, \bar{q}\, p^{2m} q^{2n} r$$

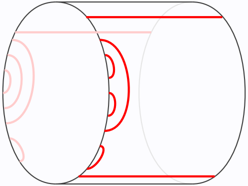
are odd quasi-polynomials of degree $2m+3$ and $2m+2n+5$, respectively.

Clean vs. unclean

You can clean arc diagrams by removing arcs parallel to the boundary.



Conversely, any arc diagram can be created by gluing cuffs to a clean one.



Fact

There are $\bar{a}\binom{b}{\frac{b-a}{2}}$ cuffs with b points on the outer boundary and a points on the inner boundary.

Clean vs. unclean (continued)

Corollary

It follows from the previous fact that

$$G_{g,n}(b_1, \ldots, b_n) = \sum_{a_i \equiv b_i} \prod_{i=1}^{n} \binom{b_i}{\frac{b_i - a_i}{2}} \times N_{g,n}(a_1, \ldots, a_n).$$

Polynomiality for $N_{g,n}$ implies

$$\begin{aligned} G_{g,n}(b_1, \ldots, b_n) &= \sum_{a_i \equiv b_i} \prod_{i=1}^{n} \binom{b_i}{\frac{b_i - a_i}{2}} \times \sum_{d_1, \ldots, d_n = 0}^{\text{finite}} C(d_1, \ldots, d_n) \prod_{i=1}^{n} \bar{a}_i(a_i)^{2d_i} \\ &= \sum_{d_1, \ldots, d_n = 0}^{\text{finite}} C(d_1, \ldots, d_n) \prod_{i=1}^{n} \underbrace{\sum_{a_i \equiv b_i} \binom{b_i}{\frac{b_i - a_i}{2}} \bar{a}_i(a_i)^{2d_i}}_{\binom{2m_i}{m_i} \times \text{quasi-polynomial in } b_i} . \end{aligned}$$

Thus, we obtain polynomiality for $G_{g,n}$.

Generatingfunctionology

Define the "generating functions"

$$\omega^G_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1,\ldots,\mu_n=0}^{\infty} G_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} x_i^{-\mu_i-1}\, dx_i$$

$$\omega^N_{g,n}(z_1, \ldots, z_n) = \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} N_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} z_i^{\nu_i-1}\, dz_i.$$

Theorem

- *If we set* $x_i = z_i + \frac{1}{z_i}$*, then* $\omega^G_{g,n} = \omega^N_{g,n}$ *for* $(g, n) \neq (0, 1)$*.*
- *The multidifferential* $\omega_{g,n}$ *is meromorphic with poles at* $z_i = -1, 0, 1$*.*

Refinement by regions

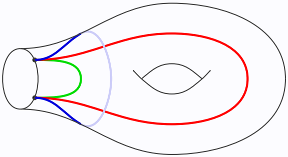
Refine the enumeration by the number of regions r and the parameter $t = r + 2g - 2 + n - \frac{1}{2}\sum b_i$.

$$G_{g,n}(b_1, \ldots, b_n) = \sum_r G_{g,n,r}(b_1, \ldots, b_n) = \sum_t G^t_{g,n}(b_1, \ldots, b_n)$$

$$N_{g,n}(b_1, \ldots, b_n) = \sum_r N_{g,n,r}(b_1, \ldots, b_n) = \sum_t N^t_{g,n}(b_1, \ldots, b_n)$$

Example

$G_{1,1,1}(2) = G^1_{1,1}(2) = 1$
$G_{1,1,2}(2) = G^2_{1,1}(2) = 2$



Theorem

- *There is a refined topological recursion for $G^t_{g,n}$ and for $N^t_{g,n}$.*
- *There are similar quasi-polynomiality results for $G^t_{g,n}$ and for $N^t_{g,n}$.*

Connections

Algebraic geometry

The leading order coefficients of $\widehat{N}_{g,n}$ satisfy

$$[x_1^{d_1} \cdots x_n^{d_n}]\, \widehat{N}_{g,n}(x_1, \ldots, x_n) = \frac{1}{2^{5g-6+2n} d_1! \cdots d_n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

These numbers are central in the celebrated Witten–Kontsevich theorem.

Mathematical physics

Topological recursions appear in various mathematical problems, many of them physically inspired.

Such problems include matrix models, Hurwitz numbers, Gromov–Witten invariants, quantum knot invariants, Chern–Simons theory, etc.

Further work: Eight more ways to count curves

Which curves to allow?

1. Arcs and non-trivial loops, where no two are parallel.
2. Arcs and non-trivial loops, where each region touches the boundary.
3. Arcs only.

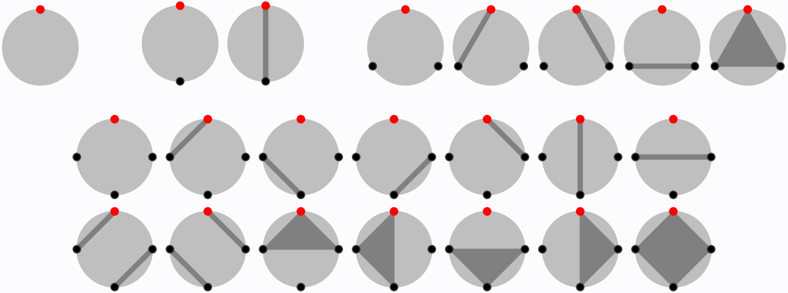
What extra conditions?

- X No conditions.
- Y Arcs are oriented so that points around a boundary alternate in/out.
- Z Arcs are oriented and regions are alternately coloured compatibly.

	X	Y	Z
1	A	B	C
2	D	E	F
3	G	H	I

Future work: Non-crossing partitions

The number of non-crossing partitions in a disk is given by the Catalan numbers: $1, 2, 5, 14, 42, \ldots$.



Question (suggested by Jang Soo Kim)

How many ways are there to partition points chosen on the boundary of a surface with disjoint "polygons"?

Selected references

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