Mathellaneous

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The Art of Tiling with Rectangles

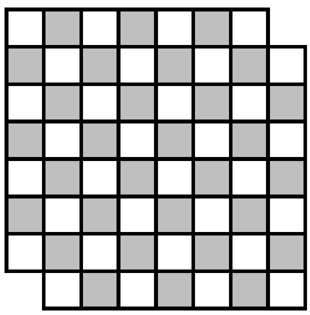
1 Checkerboards and Dominoes

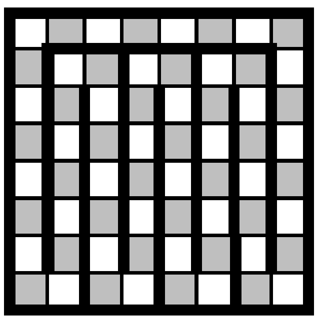
Tiling pervades the art and architecture of various ancient civilizations. Toddlers grapple with tiling problems when they pack away their wooden blocks and home renovators encounter similar problems in the bathroom. However, rather than being a frivolous pastime, mathematicians have found the art of tiling to be brimming with beautiful mathematics, problems of fiendish difficulty, as well as important applications to the physical sciences. In this article, we will consider some of the more surprising results from the art of tiling with rectangles.

One of the most famous of tiling conundrums is the following, a problem which almost every mathematician must have encountered at one time or another.

> Consider the regular 8×8 checkerboard which has been mutilated by removing two squares from opposite corners. How many ways are there to tile the remaining board with dominoes which can cover two adjacent squares?

The answer to this problem, which may seem surprising to an unsuspecting audience, is that it is impossible to tile the mutilated checkerboard. Prior to removing the two squares, there is a myriad of ways to perform such a domino tiling — actually, $3604^2 = 12988816$ ways to be precise! So why should such a trivial alteration of the board reduce this number to zero? The argument is stunning in its simplicity and the key to the solution lies in the seemingly unimportant colouring of the checkerboard into black and white squares. Of course, this colouring is such that the placement of any domino on the board will cover exactly one square of each colour. Thus, a necessary condition for the board to be tiled by dominoes is that there are an equal number of black and white squares. However, in mutilating our checkerboard, we have removed two squares of the same colour from a board that previously had 32 of each. From this disparity, we are led to the conclusion that the mutilated checkerboard cannot be tiled by dominoes, no matter how hard one might try.





From such humble beginnings, we begin our journey into the amazing world of tiling. The above well-known problem spawns a further interesting question whose answer is not quite so well-known.

Which pairs of squares may be removed from the regular 8×8 checkerboard so that the remaining board can be tiled with dominoes?

Of course, the previous argument implies that any such pair of squares must be of opposing colours. But if we remove two such squares, is it always possible to tile the remaining board with dominoes? The answer is in the affirmative and the simplest proof requires us to consider the checkerboard as a labyrinth, as pictured above. This labyrinth is hardly the design that might be used for a hedge maze, since it not only has no entrance and exit, but also consists simply of a cyclic path of 64 squares. All that is required now is to note that the removal of two squares of opposite colours divides the path now into two shorter paths, one of which may be empty. Furthermore, these two paths are of even length, so it is a trivial matter to tile them both.

Once a mathematician knows that they can do something, their next question will often be, "But in how many ways?" One of the first significant results on tiling enumeration was the following landmark theorem which was proven independently in 1961 by Fisher and Temperley [3] and by Kastelyn [6].

Theorem: The number of tilings of a $2m \times 2n$ checkerboard with dominoes is

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

One glance at the formula should be enough to indicate just how remarkable it is. The expression is the product of several terms, each one of which is usually irrational. And yet, the terms conspire together, along with the factor of 4^{mn}, to yield the answer to our enumeration problem, an integer. The interested reader may like to use this formula to verify that the number of domino tilings of the 8×8 checkerboard is indeed $3604^2 = 12988816$, as claimed earlier.

2 Tilings and Fault Lines

Thus far, we have considered only the case of tiling with 1×2 rectangles, more affectionately known as dominoes. Let us now broaden our horizons and consider the more general case of tiling with $a \times b$ rectangles, where a and b are positive integers. Of course, we can start by making the simplifying assumption that a and b are relatively prime, since other cases reduce to this after the appropriate dilation. In particular, we will discuss the following problem.

When can an $m \times n$ rectangle be tiled with $a \times b$ rectangles?

Before we state the answer, let us consider three instructive cases.

- Can you tile a 12×15 rectangle with 4×7 rectangles?
  No, of course not, since the area of each tile does not divide the area of the board.
- Can you tile a 17×28 rectangle with 4×7 rectangles?
  The answer is again in the negative, although for a more subtle reason. It turns out that 4×7 rectangles cannot even be used to tile the first column of a 17×28 rectangle. For if such a tiling is possible, we must certainly be able to write the number 17 as a sum of 4's and 7's. A quick check shows that this is not the case.
- Can you tile an 18×42 rectangle with 4×7 rectangles?
  In actual fact, it is not possible to tile the 18×42 rectangle. In fact, we will prove the stronger result that it is impossible to tile such a board with 4×1 rectangles. We will rely on a colouring argument, a generalization of the earlier proof that tiling

> a mutilated checkerboard with dominoes was impossible. In that case, the crucial feature of the natural black and white colouring of the checkerboard was the fact that each domino covered exactly one square of each colour. In the same vain, let us consider a colouring of our board such that every 4×1 rectangle placed on the board covers exactly one square of each of four colours. This can be achieved by "colouring" the square in the ith row and jth column with the colour $i + j$ modulo 4. Of course, a necessary condition for a tiling to exist is that this colouring has exactly the same number of squares of each colour. If 4 was a factor of one of the dimensions of the rectangle, then it would be clear that this condition would be satisfied. However, in this case, a simple count reveals an abundance of one colour and a deficit of another, from which we can deduce that no tiling of the 18×42 rectangle with 4×1 rectangles exists.

These arguments can be generalized to prove the following theorem, which gives a complete answer to our original problem.

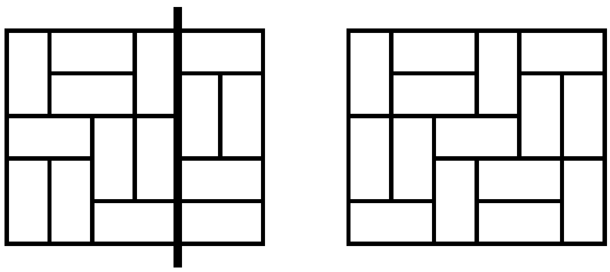
Theorem: Let a and b be relatively prime positive integers. A tiling of an $m \times n$ rectangle with $a \times b$ rectangles exists if and only if

- both m and n can be written as a sum of a's and b's; and
- either m or n is divisible by a, and either m or n is divisible by b.

Let us now turn our attention to the following beautiful tiling problem which appeared in the All Soviet Union Mathematical Olympiad back in 1963.

> A 6×6 checkerboard is tiled with 2×1 dominoes. Prove that it is possible to cut the board into two smaller rectangles with a straight line which does not pass through any of the dominoes.

Given a tiling, let us call a line which cuts the board into two pieces and yet does not pass through any of the tiles a *fault line*. For example, the diagram below shows two tilings of a 5×6 rectangle with dominoes, one which has a fault line and one which does not. This particular problem asserts that every possible domino tiling of the 6×6 rectangle must have a fault line.



In order to obtain a contradiction, let us suppose that we have a domino tiling of the 6×6 rectangle which has no fault line. Consider any one of the ten potential fault lines and, without loss of generality, we may assume that it is vertical. Since our tiling has no fault line, at least one domino must cross this vertical. However, it cannot be the only such domino, since otherwise, an odd number of squares would remain to the left of the line. Thus, this part of the board cannot be tiled with dominoes. So at least two dominoes must cross the given vertical. The same argument applies for all ten potential fault lines, so at least two dominoes must cross each of the ten potential fault lines. Since a domino may cross at most one such line, we conclude that the tiling must involve at least $10 \times 2 = 20$

dominoes. However, 20 dominoes cover an area of 40 squares, more than the area of the board in question. This contradiction implies that no faultless tiling of the 6×6 board exists.

Having solved this question, it is only natural to ask the more general question.

> When can an $m \times n$ rectangle be tiled with $a \times b$ rectangles without any fault lines?

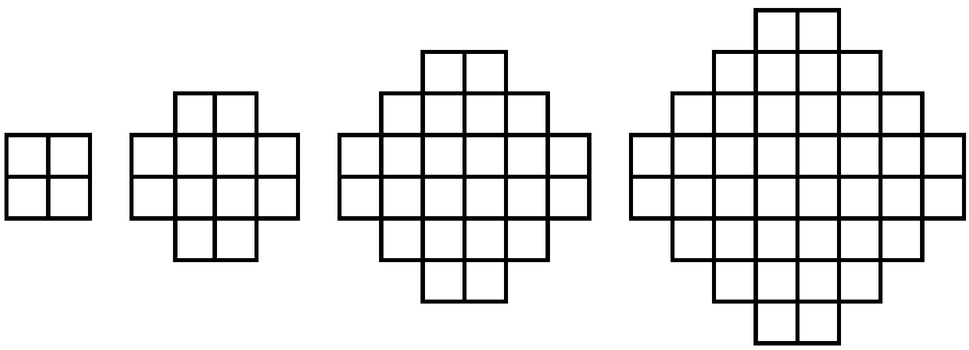
In *The Mathematical Gardner* [4], a collection of essays on recreational mathematics in honour of Martin Gardner, Ron Graham considers this exact question. Despite first appearances, there is a natural answer to this problem as described in the following theorem. Interestingly enough, the case of tiling a 6×6 rectangle with dominoes which produced such a nice mathematics competition problem, is the only exception to the rule.

Theorem: Let a and b be relatively prime positive integers. A faultless tiling of an $m \times n$ rectangle with $a \times b$ rectangles exists if and only if

- either m or n is divisible by a, and either m or n is divisible by b;
- each of m and n can be expressed as $xa + yb$ in at least two ways, where x and y are positive integers; and
- for the case where the tiles are dominoes, the rectangle is not 6×6.

3 Aztec Diamonds and Arctic Circles

Earlier, we witnessed an amazing formula enumerating domino tilings of a rectangular checkerboard. More recently, further enumeration results for domino tilings have been obtained for a skew chessboard, referred to in the literature as the Aztec diamond. The figure below depicts the Aztec diamonds of orders 1, 2, 3 and 4.

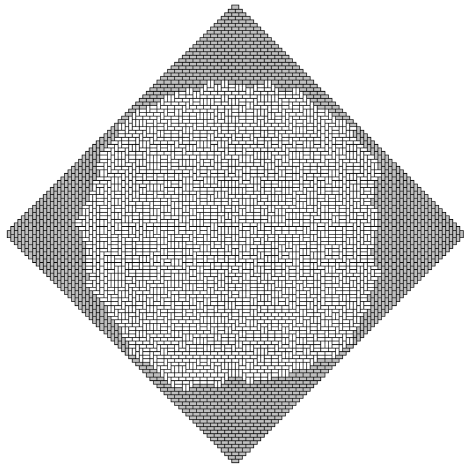


It turns out that for the four Aztec diamonds pictured above, there are precisely 2, 8, 64 and 1024 domino tilings, respectively. Of course, the astute reader will have noticed that these numbers are all perfect powers of 2. This fact is no coincidence, as verified by the following theorem.

Theorem: The number of domino tilings of the Aztec diamond of order n is $2^{\frac{n(n+1)}{2}}$.

In contrast to the checkerboard case, the enumeration formula for domino tilings of the Aztec diamond is stunning in its sheer simplicity. However, do not be fooled — the answer in no way suggests that there exists a simple proof. In fact, the result first appeared in the literature in 1992, when Elkies et al [2] demonstrated four quite involved proofs. The first exploits a connection between domino tilings and alternating-sign matrices, the second considers monotone triangles, the third uses the representation theory of $GL(n)$, while the last is more combinatorial and produces a generating function using a technique known as domino shuffling. There now exist reasonably elementary combinatorial proofs, although the constructions involved are ingeniously tricky.

However, perhaps the most amazing aspect of the Aztec diamond is not the number of domino tilings, but the nature of them. For example, consider the diagram of a particular domino tiling of the Aztec diamond of order 64. From each corner emanates an area where the domino tiling is a regular brickwork pattern. The tiles which are part of this highly organised structure have been shaded in the diagram and form what is known as the *arctic region*. On the other hand, the unshaded region is known as the *temperate zone*, and the domino tiling follows no set pattern there. Amazingly enough, it has been shown that as the order of the Aztec diamond approaches infinity, the boundary between the arctic region and the temperate zone will approach a circle for almost all of the possible domino tilings. This phenomenon is now known as the Arctic Circle Theorem and was first proved in 1998 by the team of Jockusch, Propp and Shor [5]. A more precise statement of their result is as follows.



The Arctic Circle Theorem: Let $\varepsilon > 0$. Then for all sufficiently large n, all but an ϵ fraction of the domino tilings of the Aztec diamond of order n will have a temperate zone whose boundary stays uniformly within distance εn of the inscribed circle.

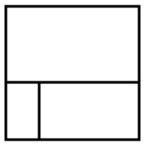
4 Tiling with Similar Rectangles

Thus far, we have restricted our attention to tilings with rectangles, all of which are congruent to each other. These have been extensively studied and the literature contains many results which involve tiling with finite sets of possible tiles. However, a quite remarkable problem arises if we broaden our horizons and consider tiles which may be of any size, but which are all similar to each other. In particular, let us consider the following problem.

> For which values of x can we tile a square with rectangles similar to the $1 \times x$ rectangle?

It is simple enough to deduce that such a tiling is possible when $x = \frac{p}{q}$ is a rational number, since a square of side length pq can be tiled by rectangles of dimensions $p \times q$. It takes a bit more consideration, however, to see that far more exotic cases can occur. For example, consider the tiling of the square with three similar rectangles shown in the diagram below. If we suppose that the small rectangle has dimensions $1 \times x$, where $x > 1$, then this forces the dimensions of the medium rectangle to be $x \times x^2$. This, in turn, implies that the large rectangle has dimensions $\left(x + \frac{1}{x}\right) \times (x^2 + 1)$. Since the whole figure fits snugly inside a square, the value of x must satisfy the equation

$$x^2 + 1 = x + \left(x + \frac{1}{x}\right) \Rightarrow x^3 - 2x^2 + x - 1 = 0.$$



To five decimal places, the unique real root of this polynomial is 1.75488. A simple consequence of the Rational Root Theorem is the fact that this particular value of x is irrational — but what other irrational values are possible? Is it possible to use transcendental numbers, such as our good friends π and e? How about the humble $\sqrt{2}$, historically significant as the first number proven to be irrational? Can we classify the values of x which allow such a tiling of the square with similar rectangles? The incredible answer to this problem was provided by Miklós Laczkovich and the late great George Szekeres [7].

Theorem: A square can be tiled by rectangles similar to the $1 \times x$ rectangle if and only if

- x is the root of an irreducible polynomial with integer coefficients; and
- the roots of this polynomial all have real part greater than zero.

Therefore, we can deduce from this theorem that it is not possible to tile a square with rectangles similar to a $1 \times \sqrt{2}$ rectangle. This is because $\sqrt{2}$ is the root of the irreducible polynomial $x^2 - 2$. Since this polynomial also has the negative root $-\sqrt{2}$, it follows from the theorem that no such tiling exists.

On the other hand, it is possible to tile the square if we take the value of x to be $\left(\frac{p}{q} + \sqrt{2}\right)$ for any rational number $\frac{p}{q} \geq \sqrt{2}$. This is because $\frac{p}{q} + \sqrt{2}$ and $\frac{p}{q} - \sqrt{2}$ are both positive and are roots of the following irreducible quadratic polynomial with integer coefficients.

$$q^2x^2 - 2pqx + (p^2 - 2q^2)$$

Here ends our brief foray into the art of tiling with rectangles. There is an abundance of fascinating tiling problems in the literature and a good starting point for the interested reader is the informative and entertaining article by Ardila and Stanley [1].

References

[1] F. Ardila, R. P. Stanley, *Tilings*, arXiv.org:math.CO/0501170, 2005.

[2] N. Elkies, G. Kuperberg, M. Larsen and J. Propp, *Alternating-sign matrices and domino tilings (Parts I and II)*, J. Alg. Combin. **1** (1992), 111–132 and 219–234.

[3] M. Fisher and H. Temperley, *Dimer problem in statistical mechanics — an exact result*, Philos. Mag., **6** (1961), 1061–1063.

[4] R. L. Graham, *Fault-Free Tilings of Rectangles*, In The Mathematical Gardner: A Collection in Honor of Martin Gardner (Ed. D. A. Klarner). Belmont, CA: Wadsworth, 120–126, 1981.

[5] W. Jockusch, J. Propp and P. Shor, *Random Domino Tilings and the Arctic Circle Theorem*, arXiv.org:math.CO/9801068, 1998.

[6] P. Kastelyn, *The statistics of dimers on a lattice I. The number of dimer arrangements on a quadratic lattice*, Phys. **27** (1961), 1209–1225.

[7] M. Laczkovich and G. Szekeres, *Tilings of the square with similar rectangles*, Discrete Comput. Geom. **13** (1995), 569–572.

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