Putnam Notes
The Year

One of the most underrated elements in the history of the development of mathematics is the notion of play. Any genuinely creative human activity has its aspect of fun; you might argue that this is where the originality comes from. But according to an expositor as straightlaced as, say, Morris Kline, this is something to be suppressed in the case of math. (Maybe, y'know, people wouldn't *take it as seriously as they should* if they knew it could be, um, taken frivolously, or made a game of.)

Enough of that. Almost all math contests that survive more than one year feature problems involving particular four-digit numbers. Frequently, the particular number (the contest year, d'accord) is almost entirely irrelevant, as in

PROBLEM 1. Suppose $S \subseteq \{1, 2, 3, \ldots, 2004\}$ is a set with 1004 elements. Show that there are two distinct numbers $a, b \in S$ such that $a + b = 2004$.

Solution: Easy one. Consider the sets $\{1, 2003\}$, $\{2, 2002\}$,...,$\{1001, 1003\}$. There are 1001 of these sets. If S has both elements of any of them, we are done. Otherwise, it has at most 1001 elements from these sets, and possibly the elements 1002 and 2004. But we assumed it had 1004 elements total. So indeed it must have both elements from one of these sets.

This is a simple consequence the pigeon-hole principle. Believe it or not, a simpler variant of this showed up on a Putnam, as an A1, of course. The point here is that 2004 could be traded in for any number (if we trade in 1004 accordingly.)

What follows is a fairly random sample of "year" problems. There are many more.

Now consider

PROBLEM 2. (B5, Putnam 1985) Find

$$\int_0^\infty t^{-\frac{1}{2}} e^{-1985(t+t^{-1})} dt.$$

[Use $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.]

Solution: Any sensible person would realize that the essence of this problem is the same if 1985 is replaced by any positive (why positive?) number α. Not being quite so sensible, we let $\alpha = \sqrt{1985}$, and we must evaluate

$$\int_0^\infty t^{-\frac{1}{2}} e^{-\alpha^2(t+t^{-1})} dt.$$

Let $t = u^{-1}$, so $dt = -u^{-2} du$, $t^{-\frac{1}{2}} = u^{\frac{1}{2}}$ and $t + t^{-1} = u + u^{-1}$; as t goes from 0 to $+\infty$, u falls from $+\infty$ to 0. With this substitution, the integral becomes

$$-\int_\infty^0 u^{-\frac{3}{2}} e^{-\alpha^2(u+u^{-1})} du = \int_0^\infty t^{-\frac{3}{2}} e^{-\alpha^2(t+t^{-1})} dt.$$

Thus the integral is

$$\frac{1}{2}\int_0^\infty (t^{-\frac{1}{2}} + t^{-\frac{3}{2}})e^{-\alpha^2(t+t^{-1})}dt.$$

Now let $v = \alpha(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. So $v^2 = \alpha^2(t + t^{-1} - 2)$; i.e., $\alpha^2(t + t^{-1}) = v^2 + 2\alpha$. $dv = \frac{\alpha}{2}(t^{-\frac{1}{2}} + t^{-\frac{3}{2}})dt$ and as t ranges from 0 to $+\infty$, v goes from $-\infty$ to $+\infty$. With this substitution, the integral becomes

$$\frac{e^{-2\alpha}}{\alpha}\int_{-\infty}^\infty e^{-v^2}dt = \frac{e^{-2\sqrt{1985}}\sqrt{\pi}}{\sqrt{1985}}.$$

A couple of comments on the handstands here. The first move ($u = t^{-1}$) was of course the last thing I did when I first solved the problem. And I tried $v = \alpha(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$ for the other substitution in the first go-round. These things usually require a couple of tries before you find what works.

One thing I've never been certain about is how careful one has to be with these indefinite integrals. (The given one is in fact indefinite at both endpoints, and should be expressed as a sum of two limits of definite integrals, as should all the subsequent ones.) I have evaluated it as an analyst would — just go ahead and justify later. I will leave the precise details of this to you.

In 1993, at least two of the problems involved the year indirectly, but I'll bet a lot of people didn't notice it in the first case.

PROBLEM 3. Identify those real numbers c such that there is a straight line which meets the graph of the curve $y = x^4 + 9x^3 + cx^2 + 9x + 3$ in four distinct points.

Solution: Note that the last two coefficients are utterly irrelevant to the question. To say this meets the graph of $y = ax + b$ in four distinct points is exactly the same thing as saying that $x^4 + 9x^3 + cx^2 + (9 - a)x + (3 - b)$ has four distinct real roots. If this happens, this function must have 3 distinct (real) relative extremums. That is, the derivative function $4x^3 + 27x^2 + 2cx + (9 - a)$ must have 3 distinct real roots. For *this* to happen, we must needs have that *its* derivative $12x^2 + 54x + 2c$ has two distinct real roots, and this implies that its discriminant $54^2 - 4(12)(2c)$ is positive, so $c < \frac{243}{8}$.

So far we have shown that $c < \frac{243}{8}$ is necessary; we now show it is sufficient. Note that the coefficient of x^2 in $(x+\frac{9}{4})^4$ is exactly $\frac{243}{8}$; letting $2d = \frac{243}{8} - c > 0$, we have that

$$x^4 + 9x^3 + cx^2 + (9 - a)x + (3 - b) = (x + \frac{9}{4})^4 - 2dx^2 + ex + f$$

for some constants e and f. By choosing a and b correctly, we can arrange that e and f are just right so that this last polynomial is

$$(x + \frac{9}{4})^4 - 2d(x + \frac{9}{4})^2 + d^2 - g^2 = [(x + \frac{9}{4})^2 - d]^2 - g^2$$

for our favourite g. This has 4 distinct roots as long as $0 < g < d$. They are $-\frac{9}{4} \pm \sqrt{d \pm g}$. We leave the rest of the details.

In some cases, the particular number *does* play a role. Consider the following (from a competition in Tartu, Estonia, in 1992).

PROBLEM 5. Does there exist a convex polyhedron with 1992 faces, all of them quadrilaterals?

Solution: Yes. First, note that if there is a convex polyhedron with n faces, all quadrilaterals, then there is one with $n + 4$ faces, all quadrilaterals. For let $ABCD$ be any face of the polyhedron Π with n faces. (We assume that the vertices A, B, C and D are arranged clockwise.) We add vertices A', B', C' and D' to Π such that the plane through A, B, C and D is parallel to that through A', B', C' and D'; and the second plane is on the opposite side of the rest of Π from the first. This can be done in such a way that if we replace $ABCD$ by the faces $AA'B'B$, $AA'D'D$, $BB'C'C$, $CC'D'D$ and $A'B'C'D'$ we still have a convex polyhedron. (A picture would help here; essentially, if $ABCD$ is on "top" of Π, you "raise" it a little, but also "shrink" it a bit, to get $A'B'C'D'$. I can be more precise, at the expense of both readability and space, but again I'm not certain how much rigour they want.)

With this observation in hand, the problem would be easy if instead of 1992, the year had been either 1990 or 1994. This is because we all know a convex polyhedron with 6 quadrilateral faces, and the last paragraph (somewhat vaguely) tells us how to construct one with $6 + 4k$ quadrilateral faces for any positive integer k.

To get 1992 faces, we need a small one to start with, such that the number of faces in the small one is divisible by 4 (like 1992). In fact, there is one with 8 quadrilateral faces. I can visualize it easily enough, but instead of describing it directly, I will describe its dual. Start with a cube and cut off the four "top" corners to the midpoints. (I.e., suppose one face of the cube is $ABCD$ in order, and the opposite face is $A'B'C'D'$ in corresponding order, the midpoints of $A'B'$, $B'C'$, $C'D'$ and $D'A'$ are E, F, G and H respectively. Our second polyhedron has vertices A, B, C, D, E, F, G and H and faces $ABCD$, $EFGH$, ABE, BCF, CDG, DAH, AEH, BEF, CFG and DGH.) Note that every vertex is on four faces in this second polyhedron.

Now if we take the dual of this polyhedron, we get one with 8 vertices, all quadrilaterals. The *dual* of a convex polyhedron Π has as its vertices the centroids of the faces of Π and an edge joins two of these centroids in the dual if and only if the corresponding faces share an edge in Π. It is well-known that the dual is also convex, the number of faces in the dual is the number fo vertices in the original (and vice versa), the number of edges in the dual is the same as in the original, and the number of edges on a face in the dual is the number of edges coming out of the corresponding vertex in the original (and vice versa).

Some comments: There may well be a more direct way to do this problem. The essential thing about 1992 was that it is divisible by 4; the construction gives us a convex polyhedron with n quadrilateral faces for any even $n \geq 6$. I don't know if there is a convex polyhedron with n quadrilateral faces for any odd n.

In "year" problems, it is often the case that the answer depends on the parity of the number; here's one that use the fact that 2002 is congruent to 1 (mod 3).

PROBLEM 6 (B4, Putnam 2002). One of the integers $1, 2, \ldots, 2002$ is selected at random – each of the them has an equal chance of being selected. You want to guess the correct answer in an *odd* number of guesses. After each guess you are told whether the secret number is greater than, equal, or less than your guess. No cheating — if you are told on some guess that the answer is bigger than m, every one of your subsequent guesses must be bigger than m. Show that you can guess in such a way so that your correct guess will fall at an odd try more than $\frac{2}{3}$ of the time.

That is, find a strategy that will win this particular game more than $\frac{2}{3}$ of the times you play it, if you play it often enough. Or more precisely, for more than $\frac{2}{3}(2002)$ choices of the mystery number, your pattern of guesses will turn up that object in an odd number of tries.

Solution: On your first guess, try $g_1 = 1$. Generally, let $g_{2m} = 3m$ and $g_{2m+1} = 3m + 1$ (where your nth guess is g_n), as long as possible. Unless you $2m$th guess is correct (bummer!), your $(2m + 1)$st guess will either be correct (hey!) or you will be able to continue unless your $2m$th guess is too high.

Eventually one of these three things must occur — you have hit the correct guess on turn $2m$ and lose, or have hit it on turn $2m+1$ since it is $3m+1$, or the guess $3m$ is too high (at turn $2m$); since you have already guessed $3(m-1)+1 = 3m - 2$ in this case, the correct answer is $3m - 1$ and you will then get at your $(2m + 1)$st turn.

So you win unless the eldricht number is $3m$ for some $m \leq [\frac{2002}{3}] = 667$. Your chance of winning is $\frac{1335}{2002} > \frac{2}{3}$.

(Incidentally, if some casino were foolish enough to institute a game like this at reasonable odds, and they found you following this strategy, they'd notice fast and kick you out. Generally, following a system gambling against a casino or the government just guarantees in the long run that you lose systematically. If they discover that your system actually *works*, they are within their legal rights to ban you.)

Sometimes the result depends on more substantial properties of the number, say its prime factorization. The most spectacular one of these I've seen is

PROBLEM 6. (Putnam 2001) Show that there is unique pair (a, b) of positive integers such that $a^{b+1} - (a + 1)^b = 2001$.

If we trade in 2001 for a different year, there may be no solutions, or many. (Examples?)

Solution: It is trivial to check that $a = 13$, $b = 2$ solves the equation. But how to find these numbers, and to show there is no other solution?

First, by the binomial theorem, $(a+1)^b \equiv 1 (mod a)$, so $a|2002$. The factors of $2002 = 2\cdot 7\cdot 11\cdot 13$ are 1,2,7,11,13,14,22,26,77,91,143,154,182,286,1001 and 2002. a must be one of these. Letting $c = a + 1$, we also have $(c - 1)^{b+1} - c^b = 2001$. If b is even, $(c-1)^{b+1} \equiv -1 (mod c)$ and $c|2002$. If b is odd, $(c-1)^{b+1} \equiv 1 (mod c)$ and $c|2000$. The factors of $2000 = 2^4 \cdot 5^3$ are 1,2,4,5,8,10,16,20,25,40,50,80,100, 125,200,250,400,500,1000 and 2000.

So either b is even and both a and $a+1$ come from the list of factors of 2002, forcing either $a = 1$ or $a = 13$; or b is odd, and a comes from that list, and $a+1$ comes from the second list — in this case a must be 1 or 7. $a = 1$ is readily seen to be impossible; we show that $a = 7$ and b odd is also out. $7^{b+1} - 8^b$ is congruent to $1 - 2 \equiv -1 (mod 3)$ if b is odd, but $3|2001$, so $a \neq 7$.

We now know that $a = 13$ and b is even. Now as $13^2 \equiv 1 (mod 8)$, we have that $13^{b+1} \equiv 13 \equiv 5 (mod 8)$. If $b \geq 3$, then $14^b \equiv 0 (mod 8)$. As $2001 \equiv 1 (mod 8)$, we must have $b = 2$.

Obviously this problem uses several particular properties of 2001 (including the factorization of the numbers above and below it!) This doesn't occur very often, but you should be on the lookout for this kind of thing.

More typically, the number just puts a bound on things. This is from Putnam 1999 (duh!).

PROBLEM 5. Show that there is a constant C such that, for any polynomial p of degree 1999,

$$|p(0)| \leq C \int_{-1}^{1} |p(x)| dx.$$

Obviously, the 1999 could be anything, but note that the constant depends on the given degree. What we really prove (by induction on n, of course) is that

For any n, there is a constant C_n such that, for any polynomial p of degree $\leq n$,

$$|p(0)| \leq C_n \int_{-1}^{1} |p(x)| dx.$$

There will not be a single constant C that works for *all* polynomials of every degree. (You see, there was this guy named Weierstrass, and he proved this theorem...)

If $n = 0$, we can obviously take $C_0 = \frac{1}{2}$. (Ditto for $n = 1$, as you should see from the first step of the general case.) Suppose we have C_m that works for each $m < n$ and we want to show that we can find such a C_n. As we will see, $C_n = 2C_m$ will work, where $m = [\frac{n}{2}]$. (How do you produce "floor"?) We assume that $n > 0$.

Note first that for any continuous g, $\int_{-1}^{1} g(x)dx = \int_{-1}^{1} g(-x)dx$. (This can be seen geometrically, or by substituting $u = -x$.) So for any p of degree $leqn$

$$\int_{-1}^{1} |p| = \frac{1}{2} \int_{-1}^{1} (|p(x)| + |p(-x)|) dx \geq \int_{-1}^{1} |q(x)| dx,$$

where $q(x) = \frac{1}{2}(p(x) + p(-x))$. Note that $q(0) = p(0)$, q has degree no larger than p, and q is even.

Thus with $m = [\frac{n}{2}]$, $q(x) = \sum_{j=0}^{m} a_j x^{2j}$ and $m < n$. Now $\int_{-1}^{1} |q| = 2\int_0^1 |q|$. We substitute $u = x^2$, $du = 2xdx$; so $dx = \frac{du}{2\sqrt{u}}$ for $u > 0$. Let $r(u) = \sum_{j=0}^{m} a_j u^j$. $\int_0^1 |q(x)|dx$ then becomes the (improper, but convergent) integral $\int_0^1 |r(u)|\frac{du}{2\sqrt{u}}$. As $\frac{1}{\sqrt{u}} \geq 1$ on $(0, 1]$, this integral is at least $\frac{1}{2}\int_0^1 |r(u)|du$.

Substituting $-u$ for u, we see that is the same as $\frac{1}{2}\int_{-1}^{0} |r(-u)|du$. That is,

$$\int_{-1}^{1} |p(x)|dx \geq \int_{-1}^{1} |q(x)|dx \geq \frac{1}{2}\int_{-1}^{1} |g(x)|dx,$$

where $g(x) = r(x)$ for $x \geq 0$ and $g(x) = r(-x)$ for $x < 0$. g may not be a polynomial, but

$$s(x) = \frac{1}{2}(g(x) + g(-x)) = \frac{1}{2}(r(x) + r(-x))$$

is. Note that $p(0) = q(0) = r(0) = g(0) = s(0)$. Also,

$$\int_{-1}^{1} |g(x)|dx = \int_{-1}^{1} \frac{1}{2}(|g(x)| + |g(-x)|)dx \geq \int_{-1}^{1} |s(x)|dx.$$

As s has degree $\leq m$, this last integral is at least $\frac{1}{C_m}|s(0)| = \frac{1}{C_m}|p(0)|$. Now $\int_{-1}^{1} |p(x)|dx \geq \frac{1}{2}\int_{-1}^{1} |s(x)|dx \geq \frac{1}{2C_m}|p(0)|$ and we finish. (Let $C_n = 2C_m$.)

In the next one, not only are the particular numbers 19 and 93 bogus (they are big enough so that you can't do the problem straight by inspection in the given case) but I found that they got in the way of finding the solution. Unlike in the previous two problems, they do have to be positive integers.

PROBLEM 5. Let $1 \leq x_j \leq 19$ for each $1 \leq j \leq 93$ and $1 \leq y_k \leq 93$ for each $1 \leq k \leq 19$. Each x_j and y_k is a positive integer, as of course are each j and k. Show that there are nonempty sets $S \subseteq \{1, \ldots, 93\}$ and $T \subseteq \{1, \ldots, 19\}$ such that $\sum_{j \in S} x_j = \sum_{k \in T} y_k$.

Solution: For each $1 \leq m \leq 19$, let $X_m = \sum_{j=1}^{m} x_j$; for $1 \leq n \leq 93$, let $Y_n = \sum_{k=1}^{n} y_k$. Clearly, each of X_{19} and Y_{93} is at most $19 \cdot 93$, and we may assume they are not equal. Suppose for the moment that $X = X_{19} > Y = Y_{93}$. Suppose also, for now, that some $x_j \neq 93$, so $X < 19 \cdot 93$.

Now consider the $19 \cdot 93$ sums $X_m + Y_n$ for all $1 \leq m \leq 19$ and $1 \leq n \leq 93$. There must be distinct pairs (m, n) and (r, s) such that $X_m + Y_n$ and $X_r + Y_s$ are equivalent $(mod X)$. Since both of these sums are $< 2X$, it must be the case that either $X_m + Y_n = X_r + Y_s$ or (WLOG) $X_m + Y_n - (X_r + Y_s) = X$.

In the first case, we must have either $m > r$ and $n < s$, or $m < r$ and $n > s$. In the first of these subcases, we have $\sum_{j=r+1}^{m} x_j = \sum_{k=n+1}^{s} y_k$. In the other subcase, $\sum_{j=m+1}^{r} x_j = \sum_{k=s+1}^{n} y_k$. Either way we're done.

Otherwise, we must have $m > r$ and $s > n$ (since both $X_m - X_r$ and $Y_n - Y_s$ are less than X). But then $\sum_{j=r+1}^{m} x_j + \sum_{k=s+1}^{n} y_k = X = \sum_{j=1}^{19} x_j$. So $\sum_{k=s+1}^{n} y_k = \sum_{j=1}^{r} x_j + \sum_{j=m+1}^{19} x_j$. (Actually, if $m = 19$, that last summand is empty, but the others are not.)

We are done unless each $x_j = 93$. Now consider the 93 sums $Y_1, \ldots, Y_{93}$. Either one of these is divisible by 93, in which case we take enough x_j's to add up to it, or two Y_n and Y_s of them are equivalent (mod 93), in which case some sum of the x_j's adds up to $Y_s - Y_n = \sum_{k=n+1}^{s} y_k$. (I assume that $n < s$.)

It looks like we should also consider the possibility that $Y > X$. (The situation is not transparently symmetric, since 19¡93.) But in that case we can just reverse the roles of x_j and y_k and of 19 and 93. Really, we could replace 19 and 93 by any positive M and N and get the same result.

Also, incidentally, we don't really need to split off the case where every $x_j = N$. But I think this is the least awkward way to present the proof. (If you can shorten and/or clarify it, let me know.)