Putnam notes//The harmonic series

Almost the first divergent series (other than something like $\sum_{n=1}^{\infty} n$) that everybody sees is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Occasionally, we see problems that are based on the proof of this fact, so we will show something slightly more general; a simple variation on this has appeared on the Putnam.

PROBLEM 1. Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, and that $a_n \leq a_{2n-1} + a_{2n}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

Of course the harmonic series satisfies this condition. Solution: We show that, for any $m \geq 1$ $\sum_{n=1}^{2^m} a_n \geq a_1 + ma_2$. This will do it, of course (by the Archimedean property of the reals). Actually, we see that $\sum_{n=2^m+1}^{2^{m+1}} a_n = \sum_{k=2^{m-1}+1}^{2^m} (a_{2k-1} + a_{2k}) \geq \sum_{k=2^{m-1}+1}^{2^m} a_k$ for any $m \geq 1$. If we assume as an induction hypothesis that this latter is $\geq a_2$, we get that each of the partial sums from $n = 2^m + 1$ to 2^{m+1} is $\geq a_2$. Since $\sum_{n=1}^{2^m} a_n = a_1 + a_2 + \sum_{n=2^1+1}^{2^2} a_n + \sum_{n=2^2+1}^{2^3} a_n + \cdots + \sum_{n=2^{m-1}+1}^{2^m} a_n$, this is at least $a_1 + ma_2$, as promised.

A slight variation, left to you:

PROBLEM 2. Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, and that $a_n \leq a_{3n-2} + a_{3n-1} + a_{3n}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

More frequently, we see problems which use the result. These can come unexpectedly. For instance,

PROBLEM 3. (B3, Putnam 1985) Let $f : \mathcal{N} \times \mathcal{N} \longrightarrow \mathcal{N}$ be a function that takes each pair of positive integers to a positive integer. Suppose that f is onto and 8-to-one; that is, for each positive integer m, there are exactly 8 pairs (j, k) such that $f(j, k) = m$. show that there is a pair (j, k) such that $f(j, k) > jk$.

This is a slight paraphrase, but exactly the same question. Hard to see how the harmonic series comes in, isn't it? But just watch. Incidentally, as we will see there are two red herring in this problem; the number 8 and the specification that f is strictly 8-to-1.

Solution: Choose N such that $\sum_{n=1}^{N} \frac{1}{n} > 8$. For each $1 \leq j \leq N$, and $k \leq \frac{N!}{j}$, if we want to have $f(j, k) \leq jk$, we must have $f(j, k) \leq N!$ obviously. If this were possible, we would have $\sum_{n=1}^{N} \frac{N!}{n} > 8(N!)$ pairs (j, k) with $f(j, k) \leq N!$ But this is impossible if each value of f occurs not more than 8 times. (Each $m \leq N!$ can only occur 8 times, giving a total of $8(N!)$ values to dole out to all those pairs.)

Or, how about

PROBLEM 4. (Putnam ?) For any integer $n \geq 3$, we let D_n be the determinant of the $(n-2) \times (n-2)$ matrix which has the entries $3, 4, \ldots, n$ on the diagonal (in that order) and all other entries 1. Does the sequence $\frac{D_n}{(n-1)!}$ converge?

Solution: As is probably no surprise in this context, D_n is in fact $(n-1)![1 + \frac{1}{2} + \cdots + \frac{1}{n-1}]$, so the answer is no.

To calculate D_n, we start row-reducing the matrix. If we label the kth row R_k and let $R_k^* = R_k - R_{k+1}$ for each $1 \leq k \leq n-3$, the resulting matrix has the same determinant as the original. R_k^* has all but two entries zero; the diagonal entry drops to $k+1$, and just to the right of that, we have $-(k+2)$. The bottom row R_{n-2} has not, as yet, changed.

Ex gratia, for $n = 6$, the original matrix is $\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 6 \end{pmatrix}$ and after these row operations, it becomes $\begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & -5 \\ 1 & 1 & 1 & 6 \end{pmatrix}$.

This is not yet upper triangular, but we row-reduce it further to systematically eliminate those 1's on the bottom row. To get rid of the first one, we replace R_{n-2} by $R_{n-2,1} = R_{n-2} - \frac{1}{2}R_1^*$. The 1 in the bottom left corner becomes a zero, the 1 next to it becomes $1 + \frac{3}{2} = 3(\frac{1}{2} + \frac{1}{3})$. The rest of the 1's and the n on the diagonal are left untouched so far.

Next replace $R_{n-2,1}$ by $R_{n-2,2} = R_{n-2,1} - (\frac{1}{2} + \frac{1}{3})R_2^*$. The first two entries in this row are zeroes, the third is now $4(\frac{1}{2} + \frac{1}{3} + \frac{1}{4})$. In the case above, we get first $\begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & -5 \\ 0 & 3(\frac{1}{2}+\frac{1}{3}) & 1 & 6 \end{pmatrix}$ and then $\begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 4(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}) & 6 \end{pmatrix}$.

We proceed like this (inductively, natch) for each $k \leq n-4$; if we have $R_{n-2,k-1}$ we replace it by $R_{n-2,k} = R_{n-2,k-1} - (\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1})R_k^*$. What occurs when we do this is that we replace the first nonzero entry of $R_{n-2,k-1}$ by zero, and the next 1 gets replaced by $1 + (k+2)(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1}) = (k+2)(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+2})$.

$R_{n-2,n-4}$ will be all zeroes, except its two right-most elements will be $(n-2)(\frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n-2})$ and n. Our last matrix above is this tage for $n = 6$. One more row operation, setting $R_{n-2,n-3} = R_{n-2,n-4} - (\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-2})R_{n-3}^*$, gives us an upper-triangular matrix, with entries 2,3,…,$n-2$ and — in the corner — $n + (n-1)(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-2}) = (n-1)(1 + \frac{1}{2} + \cdots + \frac{1}{n-1})$. The 4×4 case is $\begin{pmatrix} 2 & -3 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 5(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}) \end{pmatrix}$. Clearly, this matrix has determinant $(n-1)!(1+\frac{1}{2}+\cdots+\frac{1}{n-1})$; but our row operation shave not changed the value of the determinant.

As noted above, once we have D_n the problem becomes trivial — given that you know the harmonic series diverges.

Actually, I don't remember exactly whether the problem on the Putnam asked about the sequence $\frac{D_n}{(n-1)!}$ or about $\frac{D_n}{n!}$ / The latter sequence *does* converge, to 0 in fact, because it is $\frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) < \frac{1+\ln n}{n}$. An easy application of l'Hôpital's rule now finishes this problem.

One of the dangers of a problem like this is keeping track of the exact numbers (e.g., it's easy to think that $D_n = n!(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ or some such thing). This is partly why I carried the example along.