Putnam notes
The "evening" process

There are numerous proofs known of the Arithmetic/Geometric Mean inequality (that $\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \le \frac{x_1+x_2+\cdots+x_n}{n}$ whenever x_1, x_2,…,x_n are positive real numbers). I will start by presenting my favourite, which uses a method of proof that comes in handy for showing many inequalities of this sort.

We start with a simple lemma that includes the case $n = 2$.

LEMMA Suppose that a, b, c, d are nonnegative real numbers with $a < c \le d < b$ and $a + b = c + d$. Then $ab < cd$.

Proof: Let $A = \frac{a+b}{2} = \frac{c+d}{2}$ be the common average of the pairs $\{a, b\}$ and $\{c, d\}$. Then there are nonnegative real numbers e and f with $a = A - e$, $b = A + e$, $c = A - f$, $d = A + f$; further $f < e < A$. Now $ab = A^2 - e^2$ and $cd = A^2 - f^2$, so we are done.

How does the A/G inequality follow from this? Letting $F(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdots x_n$, it is clear that if $x = \frac{x_1+x_2+\cdots+x_n}{n}$ is the average (arithmetic mean) of the numbers, then the inequality is equivalent to saying that $F(x_1, x_2, \ldots, x_n) \le F(x, x, \ldots, x)$. (In fact, as we will see, the inequality is strict unless all the x_j's are equal to x.)

Consider the numbers x_1,x_2,…, x_n laid out in order; if all of them are equal to x, there is nothing to do. Note that if any of them is less than x, there must be at least one greater than x and vice versa. We will show the inequality is true by induction on the number of x_j's that are *not* equal to x.

An noted, the result is obvious if this number is zero, so suppose it isn't. Choose x_j which is different from x but as close as possible to it (i.e., $|x - x_j|$ is the smallest positive one of the numbers $|x - x_1|, |x - x_2|, \ldots, |x - x_n|$). Now choose x_k on the opposite side of x from x_j. So either $x_j < x < x_k$ or $x_k < x < x_j$; also $|x - x_k| \ge |x - x_j|$. Suppose that we are in the first case — the other is similar. Let $x'_j = x$ and $x'_k = x_k - (x - x_j)$. For any $i \ne j, k$ we let $x'_i = x_i$.

We have $x_j < x'_j \le x'_k < x_k$ and $x_j + x_k = x'_j + x'_k$ so $x'_j x'_k > x_j x_k$ by the lemma, and this implies that $F(x_1, x_2, \ldots, x_n) < F(x'_1, x'_2, \ldots, x'_n)$. The tuple $(x'_1, x'_2, \ldots, x'_n)$ has $x'_j = x$ (and maybe $x'_k = x$, but maybe not). In any case, by the induction hypothesis, $F(x'_1, x'_2, \ldots, x'_n) \le F(x, x, \ldots, x)$; it is important to note that trading in x_j and x_k for x'_j and x'_k does not affect the average x.

This finishes the proof, including the strict inequality. The main move in this proof is of course the replacement of the elements x_j and x_k by x'_j and x'_k, which are closer together. When we did this, the left-hand side went up, but because $x_j + x_k = x'_j + x'_k$, the right-hand side was unaltered. The induction, as is often the case with "active" proofs like this, is really saying that the procedure is iterative. By replacing the original x_j's two-by-two we get closer and closer to the the case where they are all equal to the average, and every time the left-hand side goes up.

Exchanging x_j and x_k for x'_j and x'_k is known in some circles as an "evening move", for reasons I trust are clear. ("Morning moves", for me anyway, consist mostly of stretching and yawning. For "Night Moves", consult Bob Seger.)

It should be clear that very little of the fact that $F(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdots x_n$ is used in the proof; we had to establish the lemma, which was the main part of showing that F satisfies the following

DEFINITION Suppose that $F : (\mathcal{R}^+)^n \longrightarrow \mathcal{R}$, where $\mathcal{R}$ is the set of reals, and $\mathcal{R}^+$ the set of positive reals. We say that F satisfies the *(first) evening condition* if

1. F is symmetric — that is, for any permutation σ of the set $\{1, 2, \ldots, n\}$, and any positive reals x_1,x_2,... x_n, $F(x_1, x_2, \ldots, x_n) = F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$; and
2. whenever a_1, a_2, a_3,..., a_n, a'_1 and a'_2 are positive real numbers such that $a_1 < a'_1 \leq a'_2 < a_2$ and $a_1 + a_2 = a'_1 + a'_2$, then we have that $F(a_1, a_2, a_3, \ldots, a_n) < F(a'_1, a'_2, a_3, \ldots, a_n)$.

Of course this is really a pair of conditions. In fact, the symmetry condition can be removed, at the price of stating the second, main, condition for every two places (not just the first two) and in both orders. In practice, many inequalities that are invented by the Problem People (they start out as Problem Children, I'm sure) are phrased in such a way that the relevant function F is readily identified as symmetric.

It is not necessary that the domain of F be the set of n-tuples of positive reals, although in most applications it is. In fact, it can be any convex subset of $\mathcal{R}^n$ (necessarily symmetric if the symmetry property is to hold). There are other variations of the definition, some of which will be discussed below.

The lemma above shows that the product function satisfies the evening condition. The rest of the proof can be mimicked almost verbatim to show

PROPOSITION Suppose that F satisfies the evening condition. Then for any tuple $(x_1, x_2, \ldots, x_n)$ from the domain of F, if $x = \frac{x_1+x_2+\cdots+x_n}{n}$, we have $F(x_1, x_2, \ldots, x_n) \leq F(x, x, \ldots, x)$ and equality hold only if $x_j = x$ for all j.

If we weaken the inequality in the evening condition to $F(a_1, a_2, a_3, \ldots, a_n) \leq F(a'_1, a'_2, a_3, \ldots, a_n)$, we can still conclude that $F(x_1, x_2, \ldots, x_n) \leq F(x, x, \ldots, x)$, but we may have equality without all the x_j's being the same.

Let's see a couple more examples.

PROBLEM 1. Show that, for any positive real numbers x_1,x_2,...,x_n with $n \geq 3$, we have

$$\Pi_{1 \leq j < k \leq n}(\frac{x_j + x_k}{2}) \leq (\frac{x_1 + x_2 + \cdots + x_n}{n})^{\frac{n(n-1)}{2}},$$

with equality if and only if $x_j = x_k$ for all j, k.

Solution: Clearly, the function F we consider is defined by $F(x_1, x_2, \ldots, x_n) = \Pi_{1 \leq j < k \leq n} \frac{x_j + x_k}{2}$. We show that it satisfies the evening condition. It is clearly symmetric.

So suppose that a_1, a_2, a_3,..., a_n, a'_1 and a'_2 are positive real numbers such that $a_1 < a'_1 \le a'_2 < a_2$ and $a_1 + a_2 = a'_1 + a'_2$. The factors $a_j + a_k$ are the same in both $F(a_1, a_2, a_3, \dots, a_n)$ and $F(a'_1, a'_2, a_3, \dots, a_n)$ in case j and k are both greater than 2. Also the factor $a_1 + a_2$ is replaced by $a'_1 + a'_2$ which is the same thing. What changes is that each $a_1 + a_j$ is replaced by $a'_1 + a_j$ and $a_2 + a_j$ is replaced by $a'_2 + a_j$ for $j \ge 3$.

Now $(a_1 + a_j)(a_2 + a_j) = a_1a_2 + a_j(a_1 + a_2 + a_j) < a'_1a'_2 + a_j(a'_1 + a'_2 + a_j) = (a'_1 + a_j)(a'_2 + a_j)$ for every $j \ge 3$; this uses the above lemma and $a_1 + a_2 = a'_1 + a'_2$. Since this is true for each $j \ge 3$, we have that the product of all the $(a_1 + a_j)(a_2 + a_j)$'s is strictly smaller than the product of all the $(a'_1 + a_j)(a'_2 + a_j)$'s. As all the other factors on both sides of the inequality are the same, this implies that $F(a_1, a_2, a_3, \dots, a_n) < F(a'_1, a'_2, a_3, \dots, a_n)$.

We conclude by the "evening proposition" (sounds dirty, don't it?), as the right-hand side of the given inequality is of course $F(x, x, \dots, x)$ where x is the average.

PROBLEM 2. Show that, for any positive real numbers x_1,x_2,...,x_n with $n \ge 4$, we have $\Pi_{1 \le j < k < \ell \le n}(\frac{x_j + x_k + x_\ell}{3}) \le (\frac{x_1 + x_2 + \cdots + x_n}{n})^{\frac{n(n-1)(n-2)}{6}}$, with equality if and only if $x_j = x_k$ for all j, k. Generalize!

Solution: You don't want me to have all the fun, do you?

PROBLEM 3. (A5, Putnam 1978) Show that, if $0 < x_j < \pi$ for $j = 1, \dots, n$ and $x = \frac{x_1 + x_2 + \cdots + x_n}{n}$, then $\Pi_{j=1}^{n} \frac{\sin x_j}{x_j} \le (\frac{\sin x}{x})^n$.

Solution: This problem is tailor-made for the evening process, even though the domain is not all the tuples of positive reals. (It is convex, so as we noted above, the proposition will still hold if can verify the evening condition. Convexity is necessary so we can *do* the evening.)

$F(x_1, x_2, \dots, x_n) = \Pi_{j=1}^{n} \frac{\sin x_j}{x_j}$ is clearly symmetric, and the serious part of the evening condition reduces quickly to showing that

$$\frac{\sin x_1}{x_1} \frac{\sin x_2}{x_2} \le \frac{\sin x'_1}{x'_1} \frac{\sin x'_2}{x'_2}$$

whenever $0 < x_1 < x'_1 \le x'_2 < x_2 < \pi$ and $x_1 + x_2 = x'_1 + x'_2$.

This is a bit involved, so we change notation: letting $x'_1 = a$, $x'_2 = b$ and $x'_1 - x_1 = x_2 - x'_2 = c$, we have $0 < c < a \le b < b + c < \pi$, and we have to show that

$$(*) \frac{\sin(a - c)}{a - c} \frac{\sin(b + c)}{b + c} \le \frac{\sin a}{a} \frac{\sin b}{b}.$$

We will need the fairly well-known fact that f, defined by $f(x) = \frac{\sin x}{x}$ for $0 < x < \pi$ is a strictly decreasing function (and $\lim_{x \to 0^+} f(x) = 1$). This is not difficult, and is left as an exercise.

We list several inequalities, each equivalent to the one preceding it, and to $(*)$.

$$ab \sin(a - c) \sin(b + c) \le (a - c)(b + c) \sin a \sin b.$$

$$ab\sin(a-c)\sin(b+c) \le ab\sin a\sin b - c(b+c-a)\sin a\sin b.$$

$$ab\frac{\cos(b+2c-a)-\cos(a+b)}{2} \le ab\frac{\cos(b-a)-\cos(a+b)}{2} - c(b+c-a)\sin a\sin b.$$

$$ab\frac{\cos(b+2c-a)}{2} \le ab\frac{\cos(b-a)}{2} - c(b+c-a)\sin a\sin b.$$

$$c(b+c-a)\sin a\sin b \le ab\frac{\cos(a-b)-\cos(b+2c-a)}{2}.$$

$$c(b+c-a)\sin a\sin b \le ab\sin c\sin(b+c-a).$$

$$\frac{\sin a}{a}\frac{\sin b}{b} \le \frac{\sin c}{c}\frac{\sin(b+c-a)}{b+c-a}.$$

(Note that the left-hand side of $(*)$ is the right-hand side of this baby, which we still have to show.) But $\frac{\sin a}{a} \le \frac{\sin c}{c}$ and $\frac{\sin b}{b} \le \frac{\sin(b+c-a)}{b+c-a}$ follow from the calculus fact cited above and $0 < c < a$, $0 < b+c-a < b < \pi$.

We have now verified the evening condition, and the solution of the problem (including strict inequality unless all the x_j's are the same) follows from the proposition. I might add that just saying this would probably not be acceptable to the Putnam People; the proposition is not exactly a standard result, and you would have to essentially repeat the proof. I can't be sure I have the most efficient verification of the evening condition, but I can hardly imagine doing this problem without it. (I tried. In fact I managed to get a proof more-or-less directly from convexity eventually, but I don't know if it's any easier.)

One simple variation of the evening condition arises from reversing the order of the inequalities. Thus, if F is a symmetric function defined on some convex (symmetric) $A \subseteq \mathcal{R}^n$, and whenever $a_1 < a_1' \le a_2' < a_2$ and $a_1 + a_2 = a_1' + a_2'$ we must have $F(a_1, a_2, a_3, \ldots, a_n) > F(a_1', a_2', a_3, \ldots, a_n)$ we say that F satisfies the *second (or reversed) evening condition.* There is no difficulty in altering the proof above to show that in this case, if $x = \frac{x_1+x_2+\cdots+x_n}{n}$, we must have $F(x_1, x_2, \ldots, x_n) \ge F(x, x, \ldots, x)$ with equality only if all the x_j's are the same.

For instance,

PROBLEM 4. Show that, if x_j is positive for $j = 1, \ldots, n$, then $\frac{1}{n}\sum_{j=1}^n x_j^2 \ge (\frac{1}{n}\sum_{j=1}^n x_j)^2$.

Solution: We apply reverse evening to the function $F(x_1, x_2, \ldots, x_n) = \sum_{j=1}^n x_j^2$. To show that it applies, we simply need to show that $x_1^2 + x_2^2 > (x_1')^2 + (x_2')^2$ whenever $x_1 < x_1' \le x_2' < x_2$ and $x_1 + x_2 = x_1' + x_2'$. Letting $A = \frac{x_1+x_2}{2}$ we have $x_1 = A - a$, $x_2 = A + a$, $x_1' = A - b$ and $x_2' = A + b$ with $0 \le b < a < A$. So $x_1^2 + x_2^2 = 2A^2 + 2a^2$ whereas $(x_1')^2 + (x_2')^2 = 2A^2 + 2b^2$ and we are done.

[This one can also be done as a fairly simple application of the Cauchy-Schwartz-Buniakowsky inequality.]

Generalizing this, we have the following

PROBLEM 5. Show that, if x_j is positive for $j = 1, \ldots, n$, then $\frac{1}{n}\sum_{j=1}^{n} x_j^{\alpha} \geq (\frac{1}{n}\sum_{j=1}^{n} x_j)^{\alpha}$ for any $\alpha > 1$. If, on the other hand $0 < \alpha < 1$, $\frac{1}{n}\sum_{j=1}^{n} x_j^{\alpha} \leq (\frac{1}{n}\sum_{j=1}^{n} x_j)^{\alpha}$. (In both cases, we have equality only when all the x_j's are equal.)

I won't do this; you should. Also, what if we have $\alpha < 0$?

Another variation on the "evening" principle is suggested by the following problem. Recall that the *harmonic mean* $HM(x_1, x_2, \ldots, x_n)$ of the nonzero real numbers x_1,...,x_n is defined by $(HM(x_1, x_2, \ldots, x_n))^{-1} = \frac{1}{n}\sum_{j=1}^{n} x_j^{-1}$.

A by-now-standard evening arguement establishes that the harmonic mean of any set of positive numbers is no larger than their arithmetic mean (average). But in fact the harmonic mean is always less than the geometric mean $\sqrt[n]{x_1 \cdot x_2 \cdots x_n}$. The kind of evening mentioned above is of no help in proving this, but consider

DEFINITION Suppose that $F : (\mathcal{R}^+)^n \longrightarrow \mathcal{R}$, where $\mathcal{R}$ is the set of reals, and $\mathcal{R}^+$ the set of positive reals. We say that F satisfies the *geometric evening condition* if

1. F is symmetric — that is, for any permutation σ of the set $\{1, 2, \ldots, n\}$, and any positive reals x_1,x_2,... x_n, $F(x_1, x_2, \ldots, x_n) = F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$; and

2. whenever a_1, a_2, a_3,..., a_n, a_1' and a_2' are positive real numbers such that $a_1 < a_1' \leq a_2' < a_2$ and $a_1 \cdot a_2 = a_1' \cdot a_2'$, then we have that $F(a_1, a_2, a_3, \ldots, a_n) < F(a_1', a_2', a_3, \ldots, a_n)$.

PROPOSITION Suppose that F satisfies the geometric evening condition. Then for any tuple $(x_1, x_2, \ldots, x_n)$ from the domain of F, if $g = \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$, we have $F(x_1, x_2, \ldots, x_n) \leq F(g, g, \ldots, g)$ and equality hold only if $x_j = g$ for all j.

We omit the proof, except to say that in our "evening moves" here, we replace x_j and x_k with $x_j < g < x_k$ by $\hat{x}_j$ and $\hat{x}_k$ such that $x_j x_k = \hat{x}_j \hat{x}_k$, and at least one of $\hat{x}_j$ and $\hat{x}_k$ is equal to g. This does not change the geometric mean g. The geometric evening condition, of course, assures us that each such move we make increases the value of F at the point, until we hit $F(g, g, \ldots, g)$.

Now to show that $HM(x_1, x_2, \ldots, x_n) \leq GM(x_1, x_2, \ldots, x_n)$ for positive x_j's, we apply the geometric evening condition to $F = HM$. We must check that it *does* apply, and we will do that in a minute, but first notice that if $g = GM(x_1, x_2, \ldots, x_n)$, then $HM(g, g, \ldots, g) = g$, so that this does finish the proof.

So suppose that a_1, a_2, a_3,...,a_n, a_1' and a_2' are positive, that $a_1 a_2 = a_1' a_2'$, and that $a_1 < a_1' \leq a_2' < a_2$. We must show that $HM(a_1, a_2, a_3, \ldots, a_n) < HM(a_1', a_2', a_3, \ldots, a_n)$. But $HM(a_1, a_2, a_3, \ldots, a_n) = (\frac{1}{n}\sum_{j=1}^{n} a_j^{-1})^{-1}$, so what we need to show is that $a_1^{-1} + a_2^{-1} > (a_1')^{-1} + (a_2')^{-1}$. Since $a_1 a_2 = a_1' a_2'$, this is the same as showing that $a_1 + a_2 > a_1' + a_2'$.

This is pretty easy to do directly, but let's use our first lemma one more time. Suppose instead that $a_1 + a_2 \leq a_1' + a_2'$, but still $a_1 a_2 = a_1' a_2'$, and

$a_1 < a_1' \le a_2' < a_2$. There is $a_2'' \ge a_2$ such that $a_1 + a_2'' = a_1' + a_2'$ and of course $a_1 < a_1' \le a_2' < a_2''$, so by the lemma $a_1 a_2'' < a_1' a_2'$. As $a_1 a_2 \le a_1 a_2''$ we have a contradiction.

Here we give two more examples, which use geometric evening in the opposite direction.

PROBLEM 6 (A2, Putnam 2003) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} + (b_1 b_2 \cdots b_n)^{\frac{1}{n}} \le ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{\frac{1}{n}}.$$

To show this, we first note that if any $b_i = 0$, the problem is trivial. (I leave it to the reader to figure out, in this case, when equality occurs.) So suppose that every $b_i > 0$. Dividing both sides of the inequality by $(b_1 b_2 \cdots b_n)^{\frac{1}{n}}$ and letting $x_i = \frac{a_i}{b_i}$ for each i, we get the equivalent inequality

$$(x_1 x_2 \cdots x_n)^{\frac{1}{n}} + 1 \le ((x_1 + 1)(x_2 + 1) \cdots (x_n + 1))^{\frac{1}{n}},$$

with each x_i nonnegative.

Letting $F(\bar{x})$ be the right-hand side of this inequality, we note that $F(\bar{x})$ is symmetric. We show that if $x_1 < x_2$ and we replace x_1 and x_2 by x_1' and x_2' such that $x_1 < x_1' \le x_2' < x_2$ and $x_1 x_2 = x_1' x_2'$ to get $\bar{x}' = (x_1', x_2', x_3, \ldots, x_n)$, then we have $F(\bar{x}) > F(\bar{x}')$.

[So each evening move leaves the left-hand side unchanged, and reduces the right-hand side until all the x_i's are equal to the geometric mean.]

But clearly, it is enough to show that $(x_1 + 1)(x_2 + 1) > (x_1' + 1)(x_2' + 1)$ or equivalently that $x_1 + x_2 > x_1' + x_2'$, given that $x_1 < x_1' \le x_2' < x_2$ and $x_1 x_2 = x_1' x_2'$. This is a well-known result, but I will sketch how to derive it from our very first lemma. (A direct proof in the style of our first lemma's is also quite simple.)

So suppose that $0 \le x_1 < x_1' \le x_2' < x_2$ and $x_1 + x_2 \le x_1' + x_2'$. Choose x_2'' such that $x_1 + x_2 = x_1' + x_2''$. We either have $x_1 < x_1' \le x_2'' < x_2$ or $x_1 < x_2'' \le x_1' < x_2$; either way, the lemma applies to show us that $x_1 x_2 < x_1' x_2''$. But this is $\le x_1' x_2'$, contradicting the other assumption.

PROBLEM 8 (B2, Putnam 1981) Suppose that $1 \le r \le s \le t \le 4$. Find the minimum possible value of

$$(r-1)^2 + (\frac{s}{r} - 1)^2 + (\frac{t}{s} - 1)^2 + (\frac{4}{t} - 1)^2.$$

This one generalizes like crazy; the fact that there are four terms, that the numbers are bounded by 4, that the constant subtracted is 1 — all basically irrelevant. (Of course, if you change any of these things, you will alter the number. But you won't touch the idea.)

Solution: Let's start by changing the phrasing of the problem. Let $x_1 = r$, $x_2 = \frac{s}{r}$, $x_3 = \frac{t}{s}$ and $x_4 = \frac{4}{t}$. We have that $x_1 x_2 x_3 x_4 = 4$, and all of the x_j's are between 1 and 4. We want to minimize

$$(x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 + (x_4 - 1)^2$$

given these constraints.

[Two comments, before proceeding. It goes against the grain — for most people — to change a problem with 3 free variables into one with 4. In fact, if the problem were originally stated the way I just did, a lot of folks would immediately replace x_4 by $\frac{4}{x_1 x_2 x_3}$. Sometimes this is a good idea, but not here; I hope you see that making the problem symmetric in the variables like this can also be useful. Second, I imagine that the restated problem can be tackled successfully with calculus techniques — Lagrange multipliers, say — but let's do it by evening. Oh, I trust you see that it *is* exactly the same problem.]

Anyway, with the new statement, one obvious thing to try is to let $x_1 = x_2 = x_3 = x_4 = \sqrt{2}$, giving a value of $4(\sqrt{2} - 1)^2$. (At the other extreme, if we set three of equal to 1 and the other equal to 4, we get the maximum possible, which is 9. You should see that this falls out of the proof.)

What we have to check is that if we replace x_1 and x_2 by x_1' and x_2' so that $1 \le x_1 < x_1' \le x_2' < x_2 \le 4$, and $x_1 x_2 = x_1' x_2'$, then

$$(x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 + (x_4 - 1)^2 >$$

$$(x_1' - 1)^2 + (x_2' - 1)^2 + (x_3 - 1)^2 + (x_4 - 1)^2.$$

Now it's easy. The difference between the left-hand-side and the right is

$$(x_1 - 1)^2 - (x_1' - 1)^2 + (x_2 - 1)^2 - (x_2' - 1)^2$$

$$= (x_2 - x_2')(x_2 + x_2' - 2) - (x_1' - x_1)(x_1 + x_1' - 2).$$

Clearly $x_2 + x_2' - 2 > x_1 + x_1' - 2 > 0$. But also $x_2 - x_2' > x_1' - x_1$ as $x_1 + x_2 > x_1' + x_2'$. That's it.

The basic idea of "evening" is amenable to any number of situations. We have seen examples of evening towards the arithmetic mean, and towards the geometric mean. I confess I have never seen an instance where one would naturally even things toward the harmonic mean, but it's not out of the question.