Putnam Practice Exam, 2009

1. Evaluate
$$\int_{-3}^{3} \frac{dx}{1+e^{2x^3}}.$$

2. Show that the product of 5 consecutive positive integers, the smallest of which is odd, cannot be either a perfect square or a perfect cube.

3. Let a be a real number ≥ 1; we recursively define x_n by:
$$x_0 = a, \quad x_{n+1} = a^{x_n}.$$
For which values of a does this sequence converge? Justify your answer.

4. Find every function f from the complex numbers to the complex numbers satisfying
$$f(z) + zf(1-z) = 1 + z$$
for every complex number z.

5. Suppose that $n \geq 2$ is an integer. Show that there are positive integers k and ℓ with $k \leq \ell$ such that
$$\frac{1}{n} = \sum_{j=k}^{\ell} \frac{1}{j(j+1)}.$$
For which n is the choice of k and ℓ unique? Justify.

6. Recall that a rhombus is a quadrilateral with all four sides of the same length. Suppose that you are given a triangle and you want to fit a rhombus into it so that the rhombus and the triangle share an angle, and that the rhombus has the biggest possible area. How do you do it?

   More specifically, say that the sides of the triangle are a, b and c with $a \leq b \leq c$, but $a \neq c$.

   (a) Show that you will *not* choose the angle opposite the side of length b. How do you decide which of the other two angles to choose?

   (b) Once you have chosen the angle, explain how to find the point on the opposite side that is a vertex of the rhombus.